Teaching and Learning MATHEMATICS

SOME PAST AND CURRENT APPROACHES TO MATHEMATICS EDUCATION

edited by

Laura Branchetti

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Volume 7

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Some Past and Current Approaches to Mathematics Education

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[...] *On fait la science avec des faits comme une maison avec des pierres; mais une accumulation de faits n'est pas plus une science qu'un tas de pierres n'est une maison.*

[(...) Science is built up of facts, as a house of stones. But a collection of facts is no more a science than a heap of stones is a house.]

Henri Poincaré (1908), *La Science et l'Hypothèse*.

1. Some current challenges of mathematics education

Mathematics is considered an undeniable base for the education of young people, and a source of fundamental knowledge for contemporary and future citizens. Mathematics education has attracted the attention of public opinion and policy makers in EU, so much so that "competence in mathematics has been identified at EU level as one of the key competences for personal fulfilment, active citizenship, social inclusion and employability in the knowledge society of the 21st century"[1]. This growing attention to the spread of mathematical skills is not restricted to Europe. It is in fact a worldwide trend, as

[1] European Commission/EACEA/Eurydice (2011).

Laura Branchetti, "Some Past and Current Approaches to Mathematics Education"
in Laura Branchetti, *Teaching and Learning Mathematics.*
Some Past and Current Approaches to Mathematics Education, pp. 7-19.

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shown by the increasing number of international reports on education, such as the OECD reports on PISA assessment results[2]. As this report underlines, "Proficiency in mathematics is a strong predictor of positive outcomes for young adults, influencing their ability to participate in post-secondary education and their expected future earnings"[3].

So today mathematics education arouses the interest of mathematicians, teachers, philosophers, pedagogists, psychologists, policy makers, and entrepreneurs. The current debate on mathematics education is vivid and multifaceted, but the socio-economic aspects of mathematics education are often emphasized over other no less important ones. Mathematical competence, problem solving, mathematical modeling skills for developing the most competitive technologies for industry, mathematical skills in everyday life, and computational thinking have become keywords in slogans of political schedules and international assessments[4].

As we can read above the entrance of the *Wiener Secession* building, *Der Zeit ihre Kunst, der Kunst ihre Freiheit* ("To every age its art. To art its freedom"). The prevailing attention to mathematics in function of its utility in professional performance and technology is a sort of monument to our age. Recommendations about the formative role of a *mathematics for thinking* often fall back into discourses that mirror the main trends of our society[5]. But, just like Vienna, with its rich architectural heritage, represents several periods and styles, so has mathematics education undergone many metamorphoses, changing issues, topics, methods and aims, so that today the field is characterized by a great variety of styles. Thus, looking into the past or towards the future, other social issues concerning mathematics education and new challenges are brought into light. We are also led to recognize some surprising resemblances between the approaches in different historical periods and, *mutatis mutandis,* common structures and similar concerns[6].

[2] OECD (2014).
[3] OECD (2014: 6).
[4] OECD (2014); European Commission/EACEA/Eurydice (2012).
[5] McBride (1994); Ernest, Greer & Sriraman (2009).
[6] For a complete work on the history of mathematics education, the handbook edited by Schubring (2014) is recommended.

2. Many facets of mathematics as a formative discipline

In order to broaden the horizons of this topic, let us consider, through the *dicta* of some distinguished scholars of the past, some of the possible reasons why mathematics proves invaluable in intellectual development.

Over the entrance to Plato's Academy was carved one of the most famous mottos of the ancient world: Ἀγεωμέτρητος μηδεὶς εἰσίτω ("Let no one who is ignorant of geometry enter").

Roger Bacon (1214-1294), Franciscan friar, philosopher and scientist, wrote: "Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of the world"[7].

Galileo Galilei (1564-1642), in turn, has intrigued generations with his statement: "[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word"[8]. In this respect, learning mathematics is like learning the secret language of Nature.

In the words of John Locke (1632-1704): "I have mentioned mathematics as a way to settle in the mind a habit of reasoning closely and in train"[9].

"The highest category of imaginative intellect is always eminently mathematics". Is this a claim from a famous mathematician? Not at all. It was, instead, the opinion of Edgar Allan Poe (1809-1849), vanguard of the Gothic tradition in literature and inventor of more than one literary genre[10].

Bruno D'Amore, editor and author of *Matematica, stupore e poesia* (2012) and *Arte e Matematica. Metafore, analogie, rappresentazioni, identità tra due mondi possibili* (2015), stresses the profound relationship between mathematics, other languages and forms of culture which have more traditionally been considered as pertaining to the humanities.

On closer view, some of these statements, even if to a lesser extent, are still appropriate now.

Several other reasons to teach mathematics, not necessarily good or well-founded, arise when the issue of mathematics education is faced by representatives of institutions in practice (Niss, 1996). Many beliefs and "impersonal societal forces" (ibidem), that make vague and fuzzy the goals of mathematics education, affect decisions and teaching strategies.

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[7] Kline (1969: 1).
[8] Galilei (1623: 16); English translation by the author.
[9] Locke (1706).
[10] Bagni (2000).

Niss (1996) listed some of these reasons, highlighting the difference between reasons and goals. He grouped the reasons in three macrocategories: technological and socioeconomical development, society's development and individual formation to cope with everydaylife and to prepare for future career.

The second category is described as more aleatory and susceptible of interpretations. For instance "during the last two couple of centuries, societies have often seem mathematics education as contributing to the very formation of society's political, ideological and cultural 'superstructure'" (ibidem). Some of these may sound strange, like national defence, patriotism, devotion to duty, tolerancy, cognitive independence, but are well documented by the author.

This large variety of possibilities contributes to paint a very articulated frame, that makes sometimes very hard for mathematics educations research communities to make clearly understandable their aims and the peculiarity of the research in this field, not only to external interlocutors but also within the communities of research. This obliges researchers to reflect deeply on their position and to declare it to the readers since, every time a scholar investigates teaching and learning phenomena, the milieu (Brousseu, 1986), the goals of the teacher (Schoenfeld, 2010) but also the researchers' point of view itself characterize strongly the observations. Differences at this level may also obstaculate attempts to empower teachers' efficiency or to improve teachers training programs, because the lack of an explicit agreement makes room to different implicit beliefs and personal interpretations and so compromise irremediably the communication.

3. Towards theories of mathematics teaching and learning

Leaving aside the objectives of mathematics education, another important issue has always concerned how to foster children's and adolescents' good attitude towards mathematics and how to make the learning process easier.

Until the 70's, there was only a set of mere, although eminent, opinions about mathematics education. In this field, most of the cultural resources that teachers could access were didactical materials and proposals for curricula. The enormous relevance given to such materials can be linked to the belief that learning can be changed simply by changing didactical materials. Teaching was considered, more or less implicitly, as a kind of art or, rather, "artesany" (*Ars docendi*)[11] and its efficacy entrusted solely to teachers' attitude, experience and creativity[12].

Even though the old debates have been continuing until the present day, what characterizes mathematics education today is not so much the discussion about the aims of mathematics teaching or didactical materials, but "[... it] is aimed to study the factors affecting the teaching and learning of mathematics and to develop programs to improve the teaching of mathematics"[13]. Mathematics education aims at being a science, with specific methods and theories, which has to tackle not only issues concerning the epistemology of mathematics but also its own epistemology[14]. There are multiple points of view and this is both the richness and the weakness of mathematics education as a science, or better, as a research program[15]. This problem has inspired many works, conferences, workshops and books concerning the problem of connecting theories and creating "a new conceptual space where the theories and their connections become objects of discourse and research"[16].

4. Brief description of the contents of the papers

In this volume are presented different perspectives (past and present) on mathematics education. Some examples of contemporary mathematics education theories are included and analyzed by their authors and other researchers, in order to make clearer and more comprehensible the key points of the current debates and stimulate discussion regarding future developments.

Since the complex topic of mathematics education concerns different disciplinary areas, we invited mathematicians, historian of mathematics, psychologists, philosophers and researchers on mathematics education to explore its different aspects.

An interesting approach to mathematics education, a little dated but still influential, is presented by Giorgio Bolondi. In the footsteps of Federigo Enriques, mathematician, philosopher and historian of maths, the author presents a possible point of view on the relations between mathematics, history of

[11] D'Amore (2008).
[12] D'Amore (1999).
[13] Godino, Batanero & Font (2007).
[14] Sierpinska & Lerman (1996).
[15] D'Amore & Godino (2006).
[16] Radford (2008).

mathematics and learning. In the end, he shows how this analysis can affect didactics.

The question “What *is* mathematics?” is answered by looking at how mathematics is created and how its history is reconstructed.

This last point plays a key role in Enriques’ conception of mathematics and becomes finally crucial in his theory of learning. The link between the two is logic. In fact, logic is thought of as a dynamic process rather than a static construction, i.e. a process which is based on intuition and leads the evolution of ideas. So historical reconstruction affects the inner logic of mathematics, since the way the history of mathematics is narrated structures the logic of mathematics for learners. Since reconstruction depends on posterity and is crucial for learning, teachers must be aware of this dynamic affecting the formative process.

The guiding question “What *is* mathematics?”, as an essential part of Enriques’ speculation, is a classical philosophical question, to which Marco Panza and Andrea Sereni also provide an exhaustive answer in the book *The problem of Plato*. In this volume they presents two radically opposite positions concerning the ontology of mathematical objects. The argument is carried out through the voices of the advocates of two antagonistic philosophical positions: Platonism and Nominalism. This debate provides a golden opportunity to grasp the cruxes of the ontological problem in mathematics.

Jean Dhombres takes us back to past historical periods, describing social dynamics that have led certain schools to favour some aspects of mathematics over others. Thus he awakens doubts about current similar phenomena. In his contribution, the author proposes an unusual answer to classic questions: Why does mathematics as taught in secondary schools differ from contemporary “mathematicians math”? Why must we wait so long, sometimes for several decades, to see new (though already outdated!) math at school?

The author is certainly not simply proposing to bridge the gap between didactic and contemporary mathematics: it is an impossible task that would require teachers to deal with topics that are too complex for students. Following the *fil rouge* of mathematics education trends through the last four centuries, he provides exemplar comparisons between textbook frontispieces, real textbook content and contemporary mathematics and identifies a cause of the gap between mathematics and its education in social phenomena, in what he defines “reactions”. Reactions are teachers and textbook authors’ attempts at re-writing new mathematical contents in the language of the older branches of mathematics that are still considered appropriate for education. As reactions are usually based more on current beliefs and trends rather than on solid epistemological reasons, often such reactions lead to deep transformations of the original sense ("in fact, reinventions", cit.) and underestimation of those thinking tools that are not expected to be more than operational tools for applications.

Coming back to our times, an overview on some of the mathematics education contemporary challenges is presented by Ubiratan D'Ambrosio, who has always worked to safeguard minority perspectives. All his scientific activity aims at fighting cultural oppression and illegitimate supremacies preserved by some scholastic systems. He is known all over the world for his studies in Ethnomathematics. The term was introduced for the first time by the author himself in 1977 during a presentation for the American Association for the Advancement of Science. Now, after more than three decades, he resumes part of this work in a paper concerning living matters, anything but a memory. The author intend to make this research field a forge of ideas to help us tackle the new challenges of education in a globalized world. After an overview on the aims of mathematics education in schools, the author explains the main features of the Ethnomathematics Program, with attention to qualitative thinking that has been quite lost elsewhere nowadays. The current failure to address qualitative issues is a danger for all mankind, involved as we are in the many difficult challenges requiring both qualitative and quantitative discourses. His contribution also presents many categories useful for analyzing the role of science and mathematics in society, and social issues concerning mathematics education, even though the author's own personal position on the issue is quite clear.

If we search for analogies and differences, some of D'Ambrosio's conclusions are somehow similar to those of Enriques.

The last contributions of this volume reveal a sample of the ongoing intense research activity, showing the common methods, questions and philosophical issues of this fairly recent academic research field.

Raymond Duval's semiotic theory of learning has been, and still is, very influential in our field.

He firstly summarizes his own theory, looking carefully at every detail and answering many significant theoretical questions. He then provides a detailed categorization of theories in mathematics education (defined by the French word *domaine*). He points out different criteria for characterizing theories and enumerates some possible guiding questions for developing a theory once a specific area has been selected. For example, the choice of paying more attention to one of the players in the teaching-learning process (mathematics, teacher, student, institution) is crucial. The researcher's awareness of his own point of view, i.e. of the perspective from which he/she observes the teaching-learning process, is also very important. Finally, the author points out some necessary features that a mathematics education theory should include in order to be interesting and useful.

Luis Radford presents a modern and critical analysis of his theory of objectification. In particular, the epistemological foundations of his theory are discussed from a materialistic-hegelian perspective and compared with some concrete practices. Also, at the end of his work he identifies a crux of the theory to unravel: the transformative, dialectic process in which the learners are involved cannot be grasped completely by the subject/object relationship, but must be better framed in order to take into account the process of "becoming" as related to "knowing" and *vice versa*. He proposes challenging future research questions that pave the way to interesting developments.

Bruno D'Amore carries out an accurate philological and philosophical analysis of Luis Radford's objectification theory terminology. In particular, the author (starting from the original roots of the theory) clarifies and delineates some of its keyword meanings, in order to avoid misinterpretations. Leveraging his philosophical and pedagogical expertise, the author first of all presents a dissertation on the nature of knowing and knowledge. Then he enters the core of objectification theory, examining the Marxist concept of work. He goes on to explain the fundamental role of this concept in objectification theory, similar to a sociocultural mathematics education theory based on the conception of learning as a shared practice in a group. Finally the author points out analogies between objectification theory terms and those of another theory which is usually considered very different: Brousseau's theory of situations.

Comparing these three papers, many differences stand out in terms of language and approach to mathematics education topics. Even from this handful of articles, it's clear that the panorama is articulated and widely varied.

In order to describe the contemporary situation, Anna Sfard tells us a story in dialogic form. The community of mathematics education researchers made the effort to found her discipline as a unitary science. These trials are described with the metaphor of the construction of Babel's Tower, an effective device both for describing the researchers' intentions and for helping us predict, from the very outset, the end of the story.

This article describes the current situation of mathematics education research as a scientific activity. Imagine walking down a corridor, overhearing an everyday conversation between a young PhD student and his advisor. It is a conversation wherein emerges the difficulty of finding a common ground and a common language in such a complex postmodern scenario, where every discourse has a proper status and validity. The author's stylistic choice contributes to the creation of a suspended atmosphere, possibilistic and dispersive at the same time, appropriately embodying the current common sensation. She opens up many potential approaches without actually suggesting a solution. Nevertheless, this story may become a common starting point for a new generation of young researchers, maybe a basis for new attempts at constructing the tower.

5. Brief presentation of the authors

***Giorgio Bolondi*:** After a PhD in Mathematics at the University of Nizza (France) focusing on Algebraic Geometry, his interests switched to History of Mathematics and then to Mathematics Education. Currently, he is a full professor of Mathematics and Mathematics education at the University of Bologna. He participated in the process of curricula reform in Italy and is now involved in the design and implementation of the Italian national assessment system for Mathematics.

***Laura Branchetti*:** PhD student in Mathematics Education at University of Palermo (Italy). She is currently a high school Mathematics and Physics teacher and teaches Mathematics Education at Free University of Bolzano, Faculty of Education (Italy).

Ubiratan D'Ambrosio: PhD in Mathematics at the University of São Paulo (Brazil), professor Emeritus at Universidade Estadual de Campinas/ UNICAMP (retired), ex director of the Institute of Mathematics, Statistics and Informatics (1972-1980), regular academic of the *International Academy of the History of Science.* Currently he is a professor on the Masters in Mathematics courses at University of Anhanguera, São Paulo/UNIAN and also a qualified Professor in other Brazilian universities. He received the "Felix Klein" ICMI medal and the "Kenneth O. May of History of Mathematics" ICHM medal.

***Bruno D'Amore*:** a former full professor in the Department of Mathematics University of Bologna (Italy), PhD in Mathematics Education; Phd ad honorem in Social Sciences and Education University of Cyprus. Currently he is a professor at DIE (Doctorado Interinstituciónal in Educación), enfasis matemática, at Universidad Distrital "Francisco José de Caldas", Bogotà, Colombia.

***Jean Dhombres*:** emeritus director of research at CNRS, has been working in the field of functional equations and in the theory of Banach spaces with their geometries. As professor to the School of Advanced Studies in the Social Sciences in Paris (EHESS), Jean Dhombres has examined many aspects of history of science, and some famous conflicts among scientists or about math education. He wrote various biographies on Lazare Carnot[17], Joseph Fourier and the creation of mathematical physics[18], and on the Jesuit "school of mathematics" in the Flanders during the 17th century[19]. He recently published books about Mathematics and its History, collaborating in particular in 2013 with the mathematician, awarded with the Fields medal (2010), Cédric Villani.

Raymond Duval: philosopher and psychologist, has been Professor and researcher from 1970 à 1995 at l'IREM de Strasbourg (Institut de Recherche sur l'enseignement des Mathématiques) and then at I.U.F.M. (Institut de Formation des Maîtres). His interests and research works initially dealt with systematic difficulties in learning mathematics with particular attention to the cognitive aspects of the mathematics activity and mathematical thought. His interest in the use of natural language for reasoning, distinguish mathematical and not mathematical forms of reasoning, drove him to study and finally elaborate a theory of learning based on semiotic registers of representation in mathematics, underlining the importance of coordination of different semiotic registers since it is the core of the conceptualization in mathematics.

Marco Panza: a First Class Research Director at the CNRS in History and Philosophy of Sciences, Institut d'Histoire et de Philosophie des Sciences et des Techniques, UMR of the CNRS and the University of Paris 1 and ENS Paris. He is the author of many essays and scientific papers in the field of History of Mathematics (especially regarding the Classical Age) and of Mathematics Philosophy (specifically concerning the debate on Platonism in mathematics).

Luis Radford: a full professor of Mathematics Education at Laurentian University, in Sudbury Ontario, Canada. He is Director of École des science de l'éducation. His research interests include the development of algebraic think-

[17] Fayard, 1998.
[18] Belin, 2002.
[19] *Une mécanique donnée à voir,* Brepols, 2009.

ing, the relationship between culture and thought, the epistemology of mathematics, and semiotics. He is currently working on the development of a cultural-historical theory of teaching and learning: the theory of objectification. He received the Laurentian University 2004-05 Research Excellence Award and the 2011 ICMI Hans Freudenthal Medal.

Andrea Sereni: Associate Professor at the Instiute for Advancedy Study (IUSS) in Pavia. He teaches Epistemology and Philosophy of Mathematics, and his research is also mainly focused on these areas. He has published several papers in national and international journals and collected volumes, and together with Marco Panza is the author of *Plato's Problem* (Palgrave, 2013). He is member of the COGITO, CRESA and NEtS research centres, and coordinates the promoting committee of the Italian Network for the Philosophy of Mathematics (FIlMat).

Anna Sfard: Professor of Mathematics Education at the University of Haifa. She served as Lappan-Philips-Fitzgerald Professor at Michigan State University and is a Visiting Professor in the Institute of Education, University of London, UK. Her research, focusing more generally on relations between thinking and interpersonal communication, and more specifically on the development and role of mathematical discourses in individual lives and in the course of history, has been summarized in her book, *Thinking as communicating*. She was the recipient of the 2007 Freudenthal Award.

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This volume would not have been possible without the collaboration and the suggestions of Bruno D’Amore. I especially wish to thank him for his precious contribution as an author and as a counselor. I want to thank, as well, all the authors of the chapters, who offered to the readers their great experience and competence as researchers and communicators. I want to acknowledge the work done by Massimo Sangoi, accurate and partecipating revisor. Last, but not least, I owe special thanks to Pierluigi Graziani, who has approached the new area of research with curiosity and entrusted me to edit this book.

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Epistemology and Didactics in Federigo Enriques

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Federigo Enriques (1871-1946) is a key personality of Italian culture of the first half of the XX century. One of the most prominent mathematicians of his age (the *Golden Age* of Italian Mathematics), creator with Guido Castelnuovo of the theory of algebraic surfaces, all along his life he wrote epistemological, philosophical and historical essays. He had a role in the European scientific culture of those years comparable with that of other scientists like Enrst Mach, Henri Poincaré, Wilhelm Ostwald — scholars with whom he interacted at several levels.

His contributions to the philosophy of science strongly influenced the Wiener Kreis, as Otto Neurath himself acknowledges.

He was also always concerned with educational and didactic problems, both on an institutional and on a *on-the-field* level. The central personality of Italian didactic of the post-second world war period, Emma Castelnuovo, was a pupil of his.

In this paper we will try to highlight some features of his epistemology, looking to those which were also foundations of his didactics.

1. Federigo Enriques, epistemologist

The epistemology developed by Federigo Enriques has been studied by several authors[1]. It is not an organical system, theoretically a-priori grounded in some philosophical frame. Rather, it has been a path developed in parallel — at least at the beginning — with his mathematical work. Like his engagement

[1] For further info see Castellana (2008); Pompeo, Faracovi & Speranza (1998).

Giorgio Bolondi, "Epistemology and Didactics in Federigo Enriques"
in Laura Branchetti, *Teaching and Learning Mathematics.*
Some Past and Current Approaches to Mathematics Education, pp. 21-33.

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in the study of the history of science, or in psico-cognitive questions, it can be seen as originated by his intellectual exuberance.

Since the beginning, Enriques is attracted by *cognitive* aspects of mathematical activity, and he dreams of a *scientific philosophy* which will unify all the fields of knowledge:

> La fede in questa filosofia scientifica ci ha tratto dai campi della Geometria, ove il pensiero riposa tranquillo nella sicurezza degli acquisti, a discutere sulla preparazione di una scienza gnoseologica che possa divenire oggetto d'intesa degli studiosi, e che porti ad unificare i varii dominii del sapere in una veduta sintetica del procedimento conoscitivo.[2]

This attraction was firstly catalyzed in a paper[3] which allowed him to entry in the long-term debate on the nature of Geometry — and in particular on the nature of postulates in Geometry. At the end of XIX century an animated debate took place, about the psychological roots of geometrical notions[4]. This epistemological debate resulted in many ideas and proposals about the teaching and the learning of Geometry[5]. In this paper Enriques displays a practical example of what will become one of his main epistemological principles, that we can summarize as follows: science has a *logic*, which is recapitulated in his historical development, and this *intrinsic logic* is the way for understanding how humans make, organize and learn the science itself. We underline that Enriques' arguments and examples involve advanced maths and advanced mathematical thinking, such as differential and projective geometry, and the continuum. For instance, *postulates* are essential to the inner logic of our organization of geometry, but they are also intrinsically linked to our cognition, grounded on several kind of experiences and perceptions. So, for the continuum, he writes:

> I postulati che stanno alla base della teoria del continuo costituiscono condizioni per la possibilità di unire associativamente, nei concetti della linea e della superficie, le varie rappresentazioni genetiche ed attuali che vi si collegano.[6]

In the same way, for projective geometry:

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[2] Enriques (1906).
[3] Enriques (1901).
[4] Bolondi (2002).
[5] Bolondi (1995).
[6] Enriques (1906).

> I postulati della Geometria proiettiva vengono riconosciuti come condizioni necessarie, per l'associazione di certe rappresentazioni visive, da cui hanno origine i concetti della retta e del piano.[7]

The oscillation (and sometimes the confusion) between *ontogenetic* and *phylogenetic* aspects is characteristic of Enriques' thought, and will permeate his didactics. We mention that Jean Piaget, when discussing 50 years later the *représentation de l'espace chez l'enfant*[8], explicitely refers to Enriques and to his discussion of the genesis of geometrical organization of oue experience of the space. A *trait-d'union* between Enriques and Piaget can be found in Gonseth[9].

It is clear from this that the origin of Enriques' interest in epistemology is to be found in geometry, hence in his personal research as a militant mathematician[10].

Enriques attempted, in 1906, a more general framing of these problems and debates in his book, *Problemi della Scienza*. This is the place where, starting from geometry he can also face other epistemological theories, such as Mach's empiriocriticism and Poincaré's conventionalism. In geometry, indeed, we will at the end find also his main contributions to maths education. In this book Enriques push forward the idea that the intrinsic logic of geometry is depending on our experience:

> L'ascensione del nostro intelletto dalla rappresentazione fisiologica alla geometrica, sia dovuta soprattutto al confronto dei possibili spazi visivi, nei quali sieno già introdotti gli elementi associati provenienti dai sensi diversi.[11]

This multifold relation between experience, cognition, logical organization of a science (geometry), intrinsic logic of science is at the basis of the human personal experience of science, which is hence rooted in intuition but has its specific rules.

The subsequent step of Enriques' epistemological thinking can be found in *Per la storia della Logica*[12], where he discusses the role of logic — defined as "l'insieme delle leggi che regolano un processo mentale, che solo per finzione può essere rappresentato nella forma statica d'un simbolismo",[13] in the development of scientific thinking. Enriques is a staunch opponent of the

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[7] Enriques (1898).
[8] Piaget & Inhelder (1948).
[9] Gonseth (1945); but see also Gonseth (1994).
[10] Bussotti (2006).
[11] Enriques (1906).
[12] Enriques (1922).
[13] Enriques (1922).

dominant logicism of those years, as was Poincaré from a different position. This fact — it is only by fiction that we can substitute to the logic of mental processes a static symbolism — is at the heart of his way of looking to teaching. It is not a case that a popular masterpiece of his, addressed to teachers, is titled *Insegnamento dinamico*[14]. It is worthwhile to mention that this book had an enormous widespreading in Europe; it was translated in French in 1925, in German in 1927 and in English in 1929. The German translation, performed by Ludwig Bieberbach[15] was used by Bieberbach himself in his fight against Hilbert and the Gottingen school. This is an example of how Enriques' epistemology was influential.

Later on, Enriques attached the general history of scientific thinking[16], and at the end he attempted a history of scientific knowledge[17]. For him, the problem of knowledge is the core of XX century philosophy, and scientific knowledge is the key for understanding knowledge in general. This is one of the reasons why he claims that the study of science, in school, is a highly formative task. Remember that in those decades the dominant pedagogical and educational ideology — issued from the neoidealism of Giovanni Gentile — denied any formative value to science[18]. The fight for ensuring a place in Italian schools for a *formative teaching of science,* and maths in particular, is a long-term element of Enriques' engagement in educational issues.

2. Epistemological concerns in didactics

Many mathematicians of the XXth century had deep epistemological interests, and for some of them the reflexions developed on the nature of maths became, at a certain moment, a deep and motivating component of their interest in didactics. The intertwining between epistemology, history and math activity is a characteristic of Enriques' way of working, which is also peculiar of his didactics.

Just for underlining these peculiarities, we can mention that Guido Castenuovo too, the partner of Enriques in his extraordinary math adventure — the creation of the theory of algebraic surfaces — was deeply involved in the problems of education (and of maths education in particular); he interacted with Enriques all along his public activity. But Guido Castelnuovo never

[14] Enriques (1921).
[15] Enriques (1927).
[16] Enriques & Santillana (1932); Enriques (1936).
[17] Enriques (1938).
[18] For further info see Bolondi (2000).

shared Enriques' concerns for the epistemological aspects of maths education; Castelnuovo's concerns were essentially *social* ones. In a famous passage, Castelnuovo says that:

> Ci domandiamo talvolta se il tempo che dedichiamo alle questioni d'insegnamento non sarebbe meglio impiegato nella ricerca scientifica. Ebbene, rispondiamo che è un dovere sociale che ci obbliga a trattare questi problemi. Non basta in effetti produrre la ricchezza; occorre anche procurare che la sua distribuzione avvenga senza ritardi e dispersioni. E non è forse la scienza una ricchezza, anzi la più preziosa delle ricchezze, quella che forma il nostro orgoglio e che è la fonte delle nostre gioie più pure? Non dobbiamo forse facilitare ai nostri simili l'acquisizione del sapere che è, insieme, potenza e felicità?[19]

The components of Enriques' trajectory resemble to Hans Freudenthals' ones. The starting point for Freudenthal was a reflexion on his own mathematical activity, and from this he was led to a more general philosophic reflexion on the phenomenology of maths. These reflexions involved a historical analysis which eventually led him to an engagement in research on didactics — and *Mathematics* as a whole becomes an *educational task.*[20] In Enriques, all these components and levels overlap somewhen turbulently, and we can define his didactics as the *epistemological didactics of the exuberant mathematician.*

3. The unity of Science

Since the beginning of his activity, Enriques is dreaming of a *synthesis* of sciences. When he founded *Scientia*, in 1907 — an important international journal — the chosen subtitle was *Rivista internazionale di sintesi scientifica* (international journal of scientific synthesis). Indeed, the search for tools and methods for the unity of sciences was the main goal of the journal.

He considered the problem of the classification of sciences in many articles and reviews[21]. Since science is specializing more and more, the need for unity is stronger and deeper. In a passage of *Scienza e Razionalismo*[22] we read:

> [...] Giacché un uomo solo non può cogliere ormai la totalità degli acquisti fatti, occorre almeno che i campi di azione dei lavoratori del pensiero si sovrappongano

[19] Castelnuovo (1911).
[20] Freudenthal (1973).
[21] For further info see Enriques (1910).
[22] Enriques (1912).

> e si intreccino in tutte le guise; che ciascuno proseguendo un particolare oggetto di ricerca sia indotto ad esaminarlo nella maggiore varietà dei rapporti... e così la società scientifica ritrovi — in una forma superiore — quella unità che fu condizione primitiva dell'umano pensiero.[23]

And, as Michele Ciliberto says, this unity is possible only by taking into account the processes of development of the science, and of each discipline[24]. And again: "Bisogna che tutti gli uomini illuminati inqualche ramo particolare degli studii, abbiano il sentimento dell'unità degli scopi proposti alla Scienza."[25]

His deeply rooted belief in the *unity of knowledge* is translated into a didactic principle:

> Non vi è iato o scissura fra matematiche elementari e matematiche superiori, perché queste si sviluppano da quelle, al pari dell'albero dalla tenera pianticina. E come, riguardando l'albero, potremo scoprire nella pianticina nuovi aspetti o comprendere caratteri di cui ci era sfuggito il significato, così anche lo sviluppo dei problemi matematici recherà luce sulle dottrine elementari in cui essi profondano le loro radici. Ad una condizione però: che di ogni dottrina si studi le origini, le connessioni, il divenire, non un qualsiasi assetto statico; e però che un grado di verità più alto serva ad illuminare il più basso da cui è uscito; che insomma — dopo avere studiato la scienza — ce ne valiamo per comprendere la storia. Quale modo più largo di comprensione quale più vasta esperienza didattica, che l'annodarsi dei problemi e l'urtarsi delle difficoltà entro lo spirito di tutti gli studenti, che hanno faticato prima di noi, nella scuola del mondo?[26]

4. The role of intuition in Mathematics

All along the epistemological work of Enriques there is a recurrent word: *intuition*. In his writing, this is a polysemic word, and it occurs with different *nuances* in his discussion of Poincaré's work, in his debate with Hilbert logicism, in the description of his own work as a mathematician. It is also a keyword of his educational approach to maths.

Let us quote some example, starting from Castelnuovo's description of how the theory of algebraic surfaces was built by the couple:

> Val forse la pena di accennare qual era il metodo di lavoro che seguivamo allora per rintracciare la via nell'oscurità in cui ci trovavamo. Avevamo costruito, in senso astratto s'intende, un gran numero di modelli di superficie del nostro spazio

[23] Enriques (1912).
[24] Ciliberto (1981).
[25] Enriques (1906).
[26] Enriques (1921).

> o di spazi superiori; e questi modelli avevamo distribuito, per dir così, in due vetrine. Una conteneva le superficie regolari per le quali tutto procedeva come nel migliore dei mondi possibili; l'analogia permetteva di trasportare ad esse le proprietà più salienti delle curve piane. Ma quando cercavamo di verificare queste proprietà sulle superficie dell'altra vetrina, le irregolari, cominciavano i guai, e si presentavano eccezioni di ogni specie. Alla fine lo studio assiduo dei nostri modelli ci aveva condotto a divinare alcune proprietà che dovevano sussistere, con modificazioni opportune, per le superficie di ambedue le vetrine; mettevamo poi a cimento queste proprietà colla costruzione di nuovi modelli. Se resistevano alla prova, ne cercavamo, ultima fase, la giustificazione logica. Col detto procedimento, che assomiglia a quello tenuto nelle scienze sperimentali, siamo riusciti a stabilire alcuni caratteri distintivi tra le due famiglie di superficie.[27]

Here Castelnuovo describes, indeed, Enriques' attitude towards mathematical inquire. In their letters there is a continuous dynamics between Enriques' way of looking to a problem, where intuition has a crucial role, and Castelnuovo firm anchorage to the research of formal proofs. But there is no contradiction between the two approaches: as Enriques recommend in *Insegnamento dinamico*:

> Anche la domanda consueta, se le Matematiche debbono educare piuttosto l'intuizione o la logica, è viziata per una imperfetta visione del valore dell'insegnamento. Infatti il presupposto di codesta domanda è che logica e intuizione si lascino separare come facoltà distinte dell'intelligenza, laddove esse sono piuttosto due aspetti inscindibili di un medesimo processo attivo, che si richiamano l'un altro.[28]

Already in 1904, in the preface to the *Lezioni di Geometria Proiettiva* he had exposed his didactic choices as follows:

> [...] Ho cercato di contemperare le esigenze dello spirito logico coi vantaggi e colle attrattive che l'intuizione conferisce agli studi geometrici. La traccia dello svolgimento, rigorosamente matematico, corre indipendente dalle osservazioni di carattere intuitivo, le quali dopo l'enunciazione dei postulati non sono più necessarie; ma esse compariscono tuttavia a lumeggiare alcuni concetti o ragionamenti più astrusi, ed in taluni punti possono anche sostituire con vantaggio didattico il procedimento rigoroso della dimostrazione.[29]

Note the use of terms: the *spirit of logic* has needs, whilst *intuition* has *advantages* and *attractions.* This is the didactic translation of what he had written in his review of Hilbert's *Grundlagen der Geometrie*:

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[27] Castelnuovo (1928).
[28] Enriques (1921).
[29] Enriques (1904).

> Studiando coi metodi più astratti, questioni astratte d'ordine puramente logico, l'A. mostra ovunque d'intendere, così nella scelta dei postulati di una lucida evidenza, come nelle finali applicazioni tecniche, tutto il valor dell'intuizione geometrica. Vi è in ciò un grande ammaestramento pei giovani che vogliono studiare utilmente le questioni interessanti i principi della Geometria.[30]

Indeed, the same dialectic between intuition and rigor is seen by Enriques in the historical development of mathematical thinking. When he discusses the evolution of geometrical ideas in the ancient Greece, he takes the occasion for pointing out that:

> Archimede, sia per proprio rigido criterio logico, sia per ossequio all'opinione scientifica dominante nell'ambiente accademico alessandrino, ritiene che il vero autore d'un teorema sia, non colui che per primo vi è giunto con un procedimento più o meno rigoroso, ma colui che ne ha fornito una (vera) dimostrazione, cioè una dimostrazione impeccabile. Molti matematici contemporanei sono dello stesso avviso e — non preoccupati della storia — giungerebbero, colle migliori intensioni del mondo, a spogliare per esempio i fondatori del calcolo infinitesimale, delle loro scoperte, a favore di critici, come Cauchy, Weierstrass o Dini, che due secoli più tardi vi hanno portato il rigore delle dimostrazioni.[31]

Hence for Enriques *intuition* is essential in the mathematician's path towards mathematics; *intuition* is something immanent in historical development of maths; *intuition* leads the fundamental choices before formalization; *intuition* helps in understanding the logical and the formal organization of the discipline. As a consequence, a really effective didactics cannot leave aside the intuition of the learner.

5. Federigo Enriques, historian of Science

A sentence of Enriques summarizes his position: *per noi la storia delle scienze mal si disgiunge dalla scienza.* For us, is difficult to separate the history of sciences from Science. Note the use of plural, for the history of sciences, and of the singular, for denoting The Science.

In his historical essays, the intrinsic logic of science that we mentioned above is essential for understanding the history of ideas. This methods was precious, for instance, for understanding some fragments by Parmenides that had not been understood before him, thus leading to a correct translation[32]. He advocates a *constructivist* methodology in history, opposed to a merely

[30] Enriques (1900).
[31] Enriques (1924-27).
[32] Nastasi (2008).

philological metodology. As claimed in the preface to *Storia del pensiero scientifico:*

> Costruire la storia? La domanda non ha bisogno di essere spiegata agli storici e ai filosofi; ma alla maggior parte degli scienziati occorre dire che la storia non è semplice raccolta di documenti e di testi e di notizie; che non basta possedere le fonti, ma giova intenderle, confrontandole e interpretandole secondo i criteri della logica delle idee... Soltanto per questa intelligenza [dello sviluppo scientifico e filosofico] i testi e i riferimenti acquistano il loro vero senso, alla stessa guisa delle esperienze fisiche: che non sono fatti bruti da registrare, sì anzi risposte alle aspettative teoriche, assumenti appunto il loro significato dall'ordine razionale della teoria.[33]

For Enriques, the history of science is comprehensible only at the light of the intrinsic logic of science — he has an epistemological approach to the historian's work. This intelligibility of history is a faith, for him, as the inteligibility of reality itself.

In the other direction, the history of scientific thought is a fundamental tool for the epistemologist. An anecdote: Hélène Metzger tells us a meeting of the *Centre international de synthèse* in Paris, on april, 18th 1934. Enriques is invited to speak about Parmenides, and historians like Hélène Metzger, philosophers like Henri Berr, mathematicians like Élie Cartan and André Weil are ready to listen him. Aldo Mieli, the host, introduces Enriques by saying that he is convinced that it is impossible to philosophize about sciences, and even to know sciences, without considering the history of sciences[34].

6. Enriques and the didactics of Maths

We can summarize here some points of Enriques' epistemology that we discussed in the previous paragraphs:

- Scientific knowledge has an intrinsic logic, which cannot be reduced to the formal logic of the mathematical argument
- This logic allows to read the history of science, and vice versa the history highlights our understanding of science
- Despite the specialization of sciences, we need to search for a deep unity of Science
- Mathematics activity is grounded in intuition, both for the researchers

[33] Enriques & Santillana (1932).
[34] Quoted in Nastasi (2008).

and for the learners

- Understanding mathematics needs an active role for reconstructing the intrinsic logic of science, by means also of the reconstruction of the history of mathematical ideas.

We can see these principles actualized in almost all the papers devoted by Enriques to mathematics education, and we quoted all along this paper some examples. We conclude by mentioning those that we consider the most important texts for knowing Enriques attitude and action towards didactics and mathematics education. We must mention two papers addressed to mathematics teachers: *Insegnamento dinamico*[35] and *L'errore in matematica*[36] — the last one published under nickname, due to the racial persecution in Italy. Another important document is the (already quoted) preface to the *Lezioni di Geometria Proiettiva.* Fundamental from a pactical point of view, has been his classical textbook (with Ugo Amaldi), *Elementi di Geometria*[37].

As a final quotations, we take from *Insegnamento dinamico* a sentence where the *logic* of science, in the sense of Enriques, is explained to the teachers:

> [...] Mi occorre dichiarare che la logica comprende più aspetti che di solito non si abbiano in vista dagli insegnanti di matematiche. Vi è, se così è lecito esprimersi, una logica in piccolo ed una logica in grande: intendo l'analisi raffinata del processo del pensiero esatto (quasi la veduta microscopica degli elementi che formano il tessuto della scienza) e — per contro — lo studio delle connessioni organiche del sistema, cioè la veduta macroscopica della scienza. Ora io temo che, nelle preoccupazioni dei nostri educatori matematici, la logica in piccolo tenga troppo posto in confronto alla logica in grande![38]

No need to draw explicitly consequences from this remark...

[35] Enriques (1921).
[36] Giovannini (1942).
[37] Enriques and Amaldi (1903).
[38] Enriques (1921).

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Platonism and Nominalism in Philosophy of Mathematics[1]

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1. Introduction

In a letter addressed to Thomas J. Stieltjes dated May 13th, 1894, the eminent analyst Charles Hermite writes this:

> I believe that the numbers and functions of analysis are not the arbitrary product of our minds; I think that they exist outside of ourselves with the same character of necessity as the things of objective reality; and that we encounter or discover them and that we study them as do the physicists, chemists and zoologists.[2]

In his autobiography, Godfrey Hardy, possibly the most influent number theorist of the first half of 20th century, gives voice to a similar thought:

> For me, and I suppose for most mathematicians, there is another reality which I will call 'mathematical reality'; and there is no sort of agreement about the nature of mathematical reality among either mathematicians or philosophers [...]. I believe that ~~the~~ mathematical reality lies outside us, that our function is to discover

[1] Drawn from: M. Panza and A. Sereni, *Il problema di Platone*, Carrocci Roma, 2010 (English version: *Plato's Problem*, Palgrave, Basingstoke (UK), 2013). The authors thank Laura Branchetti for her help in the preparation of the present text.
[2] Hermite and Stieltjes (1905: 398).

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> or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations.[3]

It's hard to state whether this is really the opinion of the "majority of mathematicians". Even recently, Alain Connes, winner of a Fields medal, has proclaimed :

> […] prime numbers […], as far as I'm concerned, constitute a more stable reality than the material reality that surrounds us. The working mathematician can be likened to an explorer who sets out to discover the world.[4]

And shortly after the quotation given above, Hardy adds that the same "view has been held, in one form or another, by many philosophers of high reputation from Plato onwards" (1940, p. 35)[5]. This is true beyond doubt. The idea that mathematics exists independently of human activity and lives in its own reality is shared by many mathematicians of all times. However, this is only one of different possible perspectives.

In this paper we shall articulate a brief discussion of some of these perspectives, focusing in particular on platonism and its usual alternative, namely nominalism. The study of these conceptions has some relevance to education, particularly insofar as it provides us with categories that help to account for what Speranza identify as the "implicit philosophies" and the "spontaneous epistemology" that teachers of mathematics transmit[6].

We shall begin by introducing the question that platonism is supposed to be a response of, Plato's problem, as we call it. We'll do it through a discussion of some aspect of Plato's philosophy of knowledge. We, then, present a short survey of different forms of platonism and nominalism.

2. Plato's problem

Plato's *Theaetetus* focuses on the nature of knowledge (*epistémé*). Three tentative definitions are taken into consideration and finally discarded. In Socrates' words, the verdict is that "neither perception, […] nor true opinion, nor reason or explanation combined with true opinion could be knowledge" (210a-b).

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[3] Hardy (1940: 35).
[4] Changeux, Connes (1989: 28), English translation p. 12.
[5] Hardy (1940: 35).
[6] Cf. Speranza (1997).

The thesis Plato defends in this dialogue is quite radical (in the *Meno* he does not maintain the same position): he claims that there can be no epistemology without ontology, since the first is not sufficient to itself. Let us see how he comes to this conclusion by addressing the question: What is knowledge?

Knowledge is, for Plato, knowledge of something. This something is not a state of affairs, however, but an object: we do not come to know that things are so-and-so, we know things themselves. True opinion consists then, for him, in the identification of an object with what it really is. So, Plato argues, if knowledge were justified true opinion, our true opinion would have to be combined with the interpretation (*erméneia*) of what characterizes its object, i.e. the "difference that distinguishes it". But, again, it is not sufficient that the opinion on this characterization is true; we should *know* that such a difference distinguishes the object, i.e. we should have a justified true opinion on this fact as well. Defining knowledge as justified true opinion is then clearly circular or leads to an infinite regress. In other words, Plato's point is that, even if we were able to identify an object for what it is, we could not be certain that we succeeded in doing so; thus, we could lack the appropriate reasons needed for transforming our true opinion into knowledge. Plato seems to concede that we can get true opinions on something. What he denies is that such opinions are genuine knowledge. Indeed, according to him, knowledge would require reason to make true opinion fully justified, but this reason cannot be provided. Plato's argument could be contrasted by denying that we must meet this requirement to account with reason for true opinion. If we admit that knowledge can be knowledge of something (non-propositional knowledge) as well as knowledge that something is so and so (propositional knowledge), we could maintain that we can have knowledge of *p* even without being aware of having that knowledge. In this case, however, we would renounce to bridge the epistemic hiatus Plato has highlighted in the *Theaetetus*. For, once we admit that an opinion is true only if it reproduces the reality of the idea (and the differences that characterize it), there can be no epistemology without ontology. The former appears to be insufficient to itself.

The *Meno* outlines a different view. Plato looks at an example from geometry. Socrates wisely questions Meno's servant, thereby leading him to (re)discovers how to double a square. After having recognized that true opinions are no less useful than knowledge, Socrates is then asked by Meno to explain why we need to distinguish between them. Socrates replies that

> so long as [true opinions] stay with us, [they] are a fine possession, and effect all that is good; but they do not care to stay for long, and run away out of the human soul, and thus are of no great value until one makes them fast with causal reasoning. […] But when once they are fastened, in the first place they turn into knowledge, and in the second, are abiding. And this is why knowledge is more prized than right opinion: the one transcends the other by its bonds.[7]

In addition, we are told that true opinions are learnt in a time preceding this present earthly life; reminiscence awakens them, and when they are "awakened by questioning", they "become knowledge"[8].

Hence, according to Plato, geometry stems from recollection, which brings up to the soul the true opinions that the soul had acquired before joining the body, and it does this by connecting them with each others, through a system of consequences that makes them abiding and turns them into knowledge. Thus, despite its being a earthly matter, geometry manifests knowledge, or, if you wish, is knowledge. The hiatus denounced in the *Theaetetus* is then filled here by the system of premises and consequences binding true opinions together. This system is the justification that is added to true opinion and turns it into knowledge. So, while in the *Theaetetus* justifying true opinions was required to recognize their truth, in the *Meno* it is just a means to secure them. Thus the *Meno* acknowledges the legitimacy of epistemology besides ontology, what in the *Theaetetus* was denied. While ontology is the object of true opinions (namely it is what makes them true), epistemology deals with their organization into a system. In order to find further clarification of what makes our opinions true, we must leave the *Meno* and turn, for example, to the *Phaedrus*.

In the *Phaedrus*, Plato compares souls to charioteers that conduce winged bigas to follow the gods "steeply upward to the top of the vault of heaven" until they "pass outside and take their place on the outer surface of the heaven"[9] and come to contemplate the *hyperouranios* (literally "beyond the heaven"). Here is how Plato describes what they see:

> […] the colorless, formless, and intangible truly existing essence, with which all true knowledge is concerned, holds this region and is visible only to the mind, the pilot of the soul. Now the divine intelligence, since it is nurtured on mind and pure knowledge, and the intelligence of every soul which is capable of receiving that

[7] *Meno*, 97e-98a (Translated by W.R.M. Lamb, 1967; we take the liberty to slightly modify the translations we quote when this seems to us necessary to stay closer to the original text, as we understand it).
[8] *Ibid*., 85d-86a.
[9] *Phaedrus*, 247b, in *Plato in Twelve Volumes*, Vol. 9 translated by Harold N. Fowler. Cambridge, MA, Harvard University Press; London, William Heinemann Ltd. 1925.

> which befits it, rejoices in seeing reality [*to on*] for a space of time and by gazing upon truth is nourished and made happy until the revolution brings it again to the same place.[10]

The souls who manage to follow the gods can contemplate the authentic reality while the heaven makes one of its revolutions, but, once they fall on the Earth and join to the body, they cannot preserve a memory of it. Such memory constitutes the genuine knowledge that only gods and souls before being burdened by the body can attain. The images evoked in the *Meno* and the *Phaedrus* are quite similar. Reminiscence cannot restore genuine knowledge. What it brings back are just true opinions. Owing to the impossibility of a direct contemplation of the authentic reality, only the organization of true opinions in a system can make them stable. This results in a kind of knowledge that is a copy or a trace of the genuine one. So the pessimistic view expressed in the *Theaetetus* can be reconciled with the optimistic view of the *Meno*. Justification cannot generate genuine knowledge, since this is inaccessible to us, just like the authentic reality — this is the pessimism of the *Theaetetus*. Nonetheless, organizing true opinions into a system can make them stable and produce a different sort of knowledge — this is the optimism of the *Meno*.

We are now in a position to understand what Plato says about mathematics in the *Republic* and in the *Seventh Letter*, which are generally considered as the main sources for his philosophy of mathematics.

Let us begin with the *Republic*, namely with the final part of the sixth book (509d-511e), when Socrates asks Glaucon to imagine a line divided into two unequal sections, standing respectively for the realm of the visible and that of the intelligible. For short, let us use symbols, even though Plato does not: let V be the first section of this line and J the second. In the realm of the visible we find physical bodies, their reflected images and their shadows. Thus Socrates suggests that also V gets divided into two more sections: one, V_a, stands for reflected images and shadows of physical bodies; the other, V_b, stands for these bodies themselves. Socrates adds that V_a and V_b are to each other in the same ratio as V and J, since the intelligible is to the visible as physical bodies are to their reflected images and shadows. No surprise so far. The metaphor's key is in Socrates' further suggestion: he proposes that also J gets divided according to the same ratio, and demands that its two sections do not any more stand for objects of knowledge, but rather for modes of knowledge. The first section, J_a, stands for a sort of knowledge that makes

[10] *Id*. 247c-d.

use of hypotheses, and for which what in the first division of the line are subject to imitation, namely the objects V_b, stands for, are used as images. The second section, J_b, stands for a sort of knowledge that makes no use of hypotheses since it directly appeals to ideas. Glaucon has a hard time in understanding all this. Socrates thus helps himself with an example, that of mathematics (510c-511a):

> For I think you are aware that students of geometry and logistic[11] and such subjects first postulate the odd and the even and the various figures and three kinds of angles and other things akin to these in each branch of science, regard them as known, and, treating them as hypotheses, do not deign to render any further account of them to themselves or others, taking it for granted that they are obvious to everybody. They take their start from these, and pursuing the inquiry from this point on consistently, conclude with that for the investigation of which they set out. […] they further make use of the visible forms and talk about them, though they are not thinking of them but of those things of which they are a likeness, pursuing their inquiry for the sake of the square as such and the diagonal as such, and not for the sake of the image of it which they draw […]. And so in all cases. The very things which they mould and draw, which have shadows and images of themselves in water, these things they treat in their turn as only images, but what they really seek is to get sight of those realities which can be seen only by the mind. […] This then is the class that I described as intelligible […] but with the reservation first that the soul is compelled to employ hypotheses in the investigation of it, not proceeding to a first principle because of its inability to extricate itself from and rise above its hypotheses.[12]

So geometry and logistic pertain to hypothetical knowledge. This is based on opinions which, although they are true, are justified only thanks to their organization into a system consisting of axioms, postulates, definitions, theorems, problem solutions etc. So, despite their limitations, mathematical disciplines allow us to talk about the ultimate reality of things. We cannot directly access the domain of ideal objects, nonetheless we can symbolize them by means of physical representations that hold with them the same kind of relation that reflected images or shadows hold with physical bodies.

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[11] Plato is here using the term '*logismos*'. A more literal translation would render it as 'calculation'. We prefer, however, to translate it with 'logistic' since it seems to us that Plato is here not merely referring to a mathematical procedure or practice, but rather to a branch of mathematics, namely to that branch that he elsewhere calls '*hē logistikē*' (or '*hē logistikē technē*'), a term that we shall render with 'logistic' too. Logistic was the art of calculating with numbers, and was distinct from the science about them.

[12] *Republic*, 510c-511a, in *Plato in Twelve Volumes*, Vols. 5 & 6 translated by Paul Shorey. Cambridge, MA, Harvard University Press; London, William Heinemann Ltd. 1969.

Further on[13], Socrates suggests to associate the parts of the segment with four conditions of soul: *eikasia* to V_a, *pistis* to V_b, *dianoia* to J_a, and *noēsis* to J_b. The translation of these terms continues to pose some issues to the interpreters, especially considering what Plato wrote a few lines above about geometry: "And I think you call the mental habit of geometers and their like '*dianoia*' and not '*nous*' because you regard *dianoia* as something intermediate between *doxa* and *nous*."[14] We have chosen to translate '*doxa*' as 'opinion'. Arguably '*doxa*' and '*pistis*' are almost synonymous, or at least they pertain to similar registries. On the other hand, '*noésis*' and '*nous*' share the same root, so it seems natural to take them as synonyms. The categorical shift linked to the division of the segment is reflected in the new distinction: while *eikasia* (which we could translate as "conjecture") and *doxa* or *pistis* are soul conditions (or mental states), *dianoia* and *nous* or *noésis* are faculties. Although there is no consensus on the proper translation and interpretation of this part of the dialogue, interpreters agree on the characterization of these faculties: *dianoia* is that faculty that enables us to articulate a discourse and produce arguments; *nous* is that faculty that enables us to contemplate ideas. Plato argues, then, that the opinions that are constitutive of geometry are not only true (which, however, we can only assume, since we have no direct access to ideas), but are also justified through the exercise of that special argumentative faculty which is *dianoia*. He also states that *dianoia* lies between *doxa* (producer of simple opinions) and *nous* (the faculty that allows us to contemplate ideas).

Though mathematics is, among all forms of humanly attainable knowledge, the closest to genuine knowledge, yet it does not amount to the same thing. Plato emphasizes this point in the seventh book of *Republic*:

> This at least […] will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language employed in it by its adepts. […]. Their language is ludicrous and forced, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge […] of that which always is, and not of a something which at some time comes into being and passes away.[15]

Many have seen here a claim on the ontology of geometry. This ontology would involve ideas, i.e. abstract objects, in modern terms. They are thought

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[13] *Republic*, 511d-e.
[14] *Republic*, 511d.
[15] *Republic*, 527a-b.

of as eternal, immutable and existing independently of our minds. Accordingly, Plato would here state that we can talk about the objects of geometry only by using terms that apparently refer to mere worldly objects (temporal, mutable and accessible to our senses). But, following Burnyeat[16], one could also consider that Plato is just posing a problem: rather than blaming the limitations of the language of geometry vis-à-vis the perspicuity of its ontology, he is wondering how this ontology is to be conceived of.

This interpretation has some obvious advantages. First of all, it is consistent with the distinction between genuine (*Theaetetus*) and worldly (*Meno*) knowledge. Moreover, it makes Plato's philosophy of geometry consistent with the mathematical practice of his time, a set of procedures that, a few years after Plato's death, Euclid codifies in the *Elements*. Indeed, in practice, diagrams are not only used and referred to as if they were true objects, but above all the objects of geometry, far from being treated as pure forms, are regarded as singular objects taking these forms, resulting from appropriate construction and represented by particular diagrams. The point is that geometrical practice does not directly imply a clear ontology. Hence the issue raised by Plato.

According to Burnyeat[17] , Plato would admit the following argument:

P1	Mathematical theorems are true.
P2	They are not true of physical objects in the sensible world.
[P]	———————
P3	They are true of ideal objects, distinct from sensible things.

Premise P1 would be unassailable for Plato, as he seems to have no doubts as to whether the opinions that constitute worldly knowledge are true. On the other hand, premise P2 depends on a simple observation: no physical object can verify the theorems of geometry. The conclusion is then inevitable. Another argument would then follow, this too being accepted by Plato:

P1'	Mathematicals exist.
P2'	Mathematicals are different from physical objects in the sensible world
[P']	———————
P3'	Mathematicals are ideal objects distinct from sensible things

[16] Burnyeat (1987, p. 219).
[17] Burnyeat (1987, pp. 221-222).

The conclusion P3' seems also inevitable. The problem then is to clarify the inevitable conclusions P3 and P3'. The duality between two kinds of knowledge, and hence between two aspects of geometry, underlies these two arguments. One aspect of geometry lies in its practice, in its constructive language, in the deductive system that links together its assumptions. The other aspect lies in the set of its truths. Because they are pure, they tell us about the hyperuranium. The first aspect belongs to the domain of human activity, the second to the domain of divine contemplation. The arguments P and P' provide a rationale for this duality. Another reason, perhaps even stronger, is provided in the *Seventh Letter*:

> Every existing object has three things which are the necessary means by which knowledge of that object is acquired; and the knowledge itself is a fourth thing; and as a fifth one must postulate the object itself which is cognizable and true. First of these comes the name; secondly the definition; thirdly the image; fourthly the knowledge. […] There is an object called 'circle', which has for its name the word we have just mentioned and, secondly, it has a definition, composed of names and verbs; […] And in the third place there is that object which is in course of being portrayed and obliterated, or of being shaped with a lathe, and falling into decay; but none of these affections is suffered by the circle itself […]. Fourth comes knowledge and intelligence and true opinion regarding these objects […]. And of those four intelligence approaches most nearly in kinship and similarity to the fifth, and the rest are further removed.[18]

The assumption of the existence of the fifth element, transcending the domain of geometrical practice, is here motivated by a concern for unity. Indeed, it explains why the name, the definition, the image (or diagram) and the propositions of geometry have the mutual connection we are led to ascribe to them. Plato's answer is simple: they all relate to a single object that allows us to articulate a discourse about it. On the other hand, the fourth element, be it understood as knowledge, intelligence or true opinion, represents the worldly knowledge (cf. *Memo*), and then it is related with the first three elements. Geometrical truths cannot be based on these elements, rather they depend on the fifth, that is the object itself, the idea. Truth and unity: these two requirements for earthly knowledge and geometrical practice lead to postulate an higher realm and to raise the problem of its nature.

According to this formulation, the problem seems to affect only geometry, but it actually affects also arithmetic, the other major branch of Greek mathematics. In the *Gorgias* the idea is presented in these terms:

[18] *Seventh Letter*, 342a-d, in *Plato in Twelve Volumes*, Vol. 7 translated by R.G. Bury. Cambridge, MA, Harvard University Press; London, William Heinemann Ltd. 1966.

> For instance, suppose someone asked me […]: 'Socrates, what is the art of arithmetics?' I should tell him […] that it is one of those which have their effect through speech. And suppose he went on to ask: 'With what is its speech concerned?' I should say: 'With the odd and the even, however many they are'. And if he asked again: 'What art is it that you call 'logistic'?' I should say that this also is one of those which achieve their whole effect by speech. And if he proceeded to ask: 'With what is it concerned?' I should say […] that in most respects logistic is in the same case as arithmetic, for both are concerned with the same thing, the odd and the even; but that they differ to this extent, that calculation considers the numerical values of odd and even numbers not merely in themselves but in relation to each other.[19]

Here Plato seems to distinguish (pure) arithmetic from (pure) logistic. Arithmetic, in the stricter sense, would belong to genuine knowledge, which is contemplative and inaccessible to humans, whereas logistic (distinct from numeration of particular objects) would belong to the worldly knowledge. The opinions constitutive of our worldly knowledge would owe their truth to their conformity to the objects of genuine knowledge. In the case of arithmetic, such opinions would reflect the intrinsic properties of numbers. The justification of true opinions would depend instead on a system of assumptions concerning their relations. Such assumptions are based on a multitude of particular, even physical objects (e.g. certain marks sketched on wax or clay tablets), sheer images of the undifferentiated units that constitute the numbers as such. The argument P1 could then fit the case of numbers: arithmetical theorems are true, but this is not because they are proved on the basis of arguments that display the relations between true opinions; rather this is because of their conformity to ideal objects, not to be confused with the multitude of physical objects to which calculation applies. The request for unity claimed in the *Seventh Letter* is reaffirmed here: terms, definitions and representations, as well as the propositions of arithmetic, owe their essential connection to the fact they relate to other objects, different in nature, that transcend the sphere of human practice. The problem is then the same Plato poses as regards geometry, that is understanding which is the nature of these objects. Generalized to the whole of mathematics, this is what we call 'Plato's problem'.

[19] *Gorgias*, 451a-c, in *Plato in Twelve Volumes*, Vol. 3 translated by W.R.M. Lamb. Cambridge, MA, Harvard University Press; London, William Heinemann Ltd. 1967.

3. Platonism

What is generally called 'platonism' in current philosophy of mathematics, is a family of responses to Plato's problem, all sharing the claim that mathematical statements, in particular theorems, are about abstract objects belonging to a domain that these statements describe. Philosophical views that are in may respect very far from Plato's philosophy can embed platonism so conceived.

A very common form of platonism claims the existence of the relevant abstract objects, usually referred to as 'mathematical objects', holding that mathematical theories refer to them and describe their behaviour, their properties, and their mutual relations. Actually, assuming the existence of abstract objects does not in itself contribute so much to clarify the common thesis shared by all form of platonism. In particular, this claim alone does not serve to shed light on what abstract objects are (supposed to be). On the contrary, it is by clarifying the nature of certain abstract objects that we can come to understand what claiming that they exist could mean.

Similarly, difficulties arise when we try to make sense of the claim that abstract objects, in particular mathematical objects, are abstract. There is no agreement among contemporary philosophers on how to draw the distinction between abstract and concrete objects. This might seem surprising, since this distinction is generally accepted as crucial. Focusing on particular examples, such as propositions, sets, or natural numbers — non-negative (finite) integers: 0, 1, 2, 3, ... — has led some philosophers to think that this distinction is needed to account for certain phenomena. But when we turn to the question of how the notion of abstract object is to be understood in general, we have to deal with many differences these particular examples involve, and doubts arise. Nonetheless, several attempts have been made to clarify the concept of abstract object in general. Among them, Dummett (1973), Zalta (1983) and Hale's (1987) offer noteworthy inquiries. In a general overview, it is preferable leaving to each philosopher who has defended platonism the duty of spelling out his own conception of abstract objects, rather than upholding and presenting only one solution. Suffice it to say that different ways of articulating and defending platonism give rise to various conceptions. But the issues of platonism do not end here. Indeed, one is led to ask on which grounds an object, in particular an abstract object, owns the status of mathematical object.

Answering this question would require a characterization of the relevant objects, then called 'mathematical'. The difficulty could be avoided by claiming that it is not among the task of a philosopher of mathematics that of establishing in general what has to count as mathematics, or what the nature of the objects of mathematics should be. His/her task would rather be that of accounting for certain accepted mathematical theories and possibly trying to ponder whether they could be conceived of as descriptions of particular domains of objects. A positive answer would *ipso facto* make such a philosopher count as a platonist. This answer would include a characterization of the relevant objects; these will then not be said ‘mathematical’ because they satisfy a given criterion of mathematicality, but simply because it will be established that a given theory, generally considered as mathematical, is about them.

This line of reasoning leads to regard mathematics as something subsisting prior to philosophy of mathematics, as a fact that philosophy of mathematics is called to account for. Even though many platonist philosophers think that these objects pre-exist to mathematical theories (just as trees pre-exists to botanic), they usually let mathematicians identify the objects of these theories. This way of thinking seems very natural. In fact, philosophy of mathematics has almost always been presented as a reflection on parts of mathematics that had already been constituted. And even when philosophers have tried to provide a foundation of mathematics, and to this end have suggested new definitions of certain mathematical objects or new formulations of certain theories, they have conceded, or better required, that their definitions and formulations yield the same results that had previously been obtained within mathematic.

Now, if this is the case, philosophers have the possibility of endorsing platonism limited to some mathematical theories, while refusing to take this approach or simply suspending judgment for other theories. A relevant case concerns arithmetic. In this case, it is claimed that statements of arithmetic are concerned with a special domain of abstract objects, namely natural numbers. This is a widespread form of platonism, which is particularly relevant for both the centrality of arithmetic within mathematics and its broad applications. It is widespread because, for reasons that cannot be made clear here, arithmetic is well suited to be interpreted as a theory about certain abstract objects. This is why rarely philosophers have adopted platonism for a mathematical theory and not for arithmetic. In other words, virtually all platonists about mathematics seem to be platonists about arithmetic as well, whereas many platonists on arithmetic avoid taking position as regards other fields of mathematics or — as Gottlob Frege did regarding geometry — take different approaches for some of them. Hence arithmetical platonism can be taken as a paradigmatic case.

There are good reasons for doubting that the platonist view is correct. For instance, one might suspect that its consequences are unacceptable or that it is not possible to properly justify it. Nonetheless, one cannot avoid recognizing the fascination of this perspective, which provides very simple answers to Plato's problem, together with many complex questions related to it, especially as regards the nature of mathematics: despite its shortcomings when it comes to clarify what abstract objects are and make sense of their existence, the idea that mathematical theories dealt with abstract objects allows us to regard these theories as structurally similar to other forms of discourse concerning concrete objects.

The simplicity of arithmetical Platonism might prove beneficial even to arithmetic itself. For example, we could argue that by stating that 2 is less than 3, we claim that a particular relation (the relation of being less than) holds between two abstract objects of the same kind (the numbers 2 and 3). The statement '2 < 3' would then be interpreted so that its syntactic structure faithfully reflects its semantic structure. It would not mean, for example, that in the act of calculating one must count up to two units before counting the third, but, more simply, that the object 2 is less than the object 3. Moreover, we could think that this statement is true or false regardless of its justification, that is regardless of our ability (and *a fortiori* of the procedure we could devise) to prove it. However, if we managed to prove it, this would vindicate its truth, along with the falsity of the statements that contradict it. So the fascination of arithmetical platonism depends on its simplicity and on the fact that this interpretation immediately accounts for the sense of inderogability we feel vis-à-vis mathematics and its theorems, which we are naturally led to regard as truth.

However, there is an important point to be made for the sake of precision. At some point the platonist interpretation of '2 < 3' takes a leap. In fact, claiming that this statement is about certain abstract objects is not the same as claiming that these objects exist and this statement is true. Moreover, claiming that '2 < 3' is true does not imply that its truth is independent of our ability to prove it.

What is quite commonly called platonism *tout court*, i.e. the claim that mathematical statements are about abstract objects and that such objects do exist, should be, then, more properly referred to as 'ontological realism'. This view should not be confused with the so-called 'semantic realism', which simply claims that the theorems of mathematics are true. Accepting semantic realism does not require assuming the existence of mathematical objects to ground the claim that theorems are true. Furthermore, semantic realism is in turn to be distinguished from a third view, highlighted in particular by Michael Dummett, which holds that the truth value of certain statements is independent of our ability to ascertain it. This is a version of the philosophical view usually called ‘realism’ (without further qualifications). The term ‘realism’ is often used in philosophy of science also to indicate a fourth perspective, called ‘scientific realism’.

The foregoing considerations allow us now to recognize some weaknesses of platonism. One of them lies in the lack of convincing arguments independent of both its explanatory advantages and other assumptions that are at least as strong as platonism itself. Platonism is often vindicated, indeed, based on a negative argument which goes like this: unless we accept platonism, there is no way to account for certain phenomena peculiar to mathematics. One may reasonably be unsatisfied with this kind of argument. But there is more. The way the advocates of platonism describe the access to the abstract objects domain displays a sort of religious attitude. Although for believers faith provides the way to get in touch with God and to give content to their prayers, philosophy should not settle for a similar way to attain mathematical objects. We should then ask: how do we access such domain? Answering this question is hard and this difficulty highlights another weakness of platonism.

4. Nominalism: different ways to reject platonism

Many philosophers reject platonism. This is because of its weaknesses and of other independent reasons. The alternative view, usually called ‘nominalism’, refuses to admit either that mathematics deals with abstract objects, or that it describes them, in most cases simply because no such object exist.

Some of those philosophers propose a radical or comprehensive form of nominalism. They argue that there are no abstract objects of any kind and that it is even not plausible at all to think that such abstract objects exist. Yet, one might hold them to account of the commonsense insight that there are certain things that could hardly be taken to be concrete objects. An example is the Italian national football team. Is there really no plausible sense in claiming that the Italian national football team exists, whereas the Italian national racing team does not? In other words, the burden of explaining why, for example, innocuous statements such as ‘the Italian national football team won the World Cup’ are intelligible shifts to the advocates of radical nominalism. A well-known argument is that such statements are actually appropriate abridged paraphrases of longer statements and concern concrete objects. When we say that the Italian national football team won the World Cup, we actually refer to the fact that some individuals — who, by signing contracts, are committed to each other and to other people, and are recognized by a wider community of people —, by interacting with other individuals — they also bound by similar contracts — in appropriate circumstances and according to certain rules, have obtained the highest score among all the opponents. In many cases the paraphrase could be lengthy and cumbersome, but in fact it makes explicit what we really mean by uttering the original statements. The basic idea then goes like this: even though syntax might suggest that some statements are about abstract objects (e.g. football teams), their proper interpretation leads us to recognize that they actually refer to concrete, ontologically innocent objects.

We might infer that mathematical statements can be treated the same way. In the normal form, mathematical statements would be abridged reformulations of longer and complex statements referring to concrete objects. Paraphrase involves reinterpretation. Nominalists who adopt this solution suggest then to interpret the language of mathematics in such a way as to avoid evoking abstract mathematical entities. It remains possible to express assertions in the usual way, for example in terms of numbers, classes, functions etc., but this is regarded as a practical expedient.

What alternatives are there for nominalists who think that this solution is not viable? They can follow platonism in believing that mathematical statements, even taken literally, i.e. according to their syntactic structure, are fully intelligible, but deny that they actually deal with abstract objects.

A first argument nominalists could put forward is that such statements, though perfectly intelligible, are devoid of content. They do not describe anything properly, then they do not refer to any object. Instead, they can provide an easy way to stipulate linguistic conventions, or result from such conventions, or be understood as logical truths, like the assertion 'either the sun is covered by clouds or it is not covered by clouds', which certainly is not about today's weather, nor, in spite of appearances, about the sun and the clouds.

A second alternative version of nominalism boils down to the idea that mathematical statements, taken literally, despite appearances, deal properly and exclusively with concrete objects. The statement '$2 < 3$', for example, is concerned with the objects 2 and 3, and these are just as concrete as the stars and the sun. This proposal differs from platonism only insofar as it recognizes to concrete objects the role that platonists assigns to abstract objects. Some might argue that this is nothing but a version of platonism, a sort of empiricist platonism.

Whether or not this is a form of platonism, the fact remains that the argument has often been put forward. According to John Stuart Mill (1843), for example, numbers are nothing but properties of collections of concrete objects and mathematical statements do nothing more than expressing mere empirical generalizations. Penelope Maddy (1990a, 1990b) has called 'physicalist platonism' her conception of sets, which is similar in spirit to Mill's view. And Hartry Field (1980) seems to suggest that elementary synthetic geometry (i.e. geometry as Euclid exposed it in the *Elements*, without calling into question real numbers or sets) deals with points in space-time taken as concrete objects. Apart from the motivations and plausibility of each of these views, they make apparent the possibility to distinguish, at least in principle, two claims that are usually conflated: on the one hand, that (a) mathematical statements are concerned with a domain of objects; and, on the other hand, that (b) they are concerned with abstract objects. Some arguments in favor of platonism (including the Indispensability Argument, to which we shall came later) in fact underpin (a) without supporting (b).

However, not all empiricist views in philosophy of mathematics are committed to the idea that mathematical statements (paraphrased or not) are about concrete or empirical objects. There are also those who, while rejecting this claim, maintain a form of empiricism limited to the issue of justification. For example, one might argue, as Charles Parsons (2008), that in order to study abstract objects mathematicians must handle concrete objects that hold certain complex and relevant relations with them. Other versions of empiricism require that a proper justification of claims about abstract mathematical objects is, or at least should be, of the same type as the one we apply to claims about concrete objects. This idea has been promoted by Imre Lakatos (1976) and nowadays is supported, although with different argumentations, by all those who defend the Indispensability Argument.

In the third place, advocates of nominalism could also argue that, if we admit that there are no abstract objects, then mathematical statements, which are to be taken literally, are not true or, if they are true, this is only in a vacuous way. Consider these two statements:

a) There is a prime even number.
b) All prime numbers are odd.

Let us assume that:

i) Mathematical statements are to be taken literally.

ii) There is only one sense in which we can say that these statements are true, and this is the same sense in which we usually ascribe truth to certain statements of ordinary language[20].

iii) If natural numbers existed, they would be abstract objects, but there are no such abstract objects.

Let us also assume that, as it seems natural, the syntactic structure of these statements is preserved in the following reformulations into the predicate logic language:

a') There is an *x* that is a prime number, and that is even.

b') For all *x*, if *x* is a prime number, then *x* is odd.

From condition (i) it follows that the appropriate semantic interpretation of (a) and (b) is that which is preserved in the syntactic form of (a') and (b'). We should then conclude that (a) is not true, while (b) is true. Take (a): based on condition (iii), there are neither prime nor even numbers, then *a fortiori* there is no *x* that is a prime number. Conditions (i) and (ii) entail, then, that (a) is not true. The case of assertion (b) might appear trickier. Just notice that, since (iii) posits that there is no *x* that is a prime number, from (ii) it follows that the antecedent "*x* is a prime number" is false, whatever *x*. Then condition (i) entails that the implication (i.e. the entire statement) is true. Indeed, according to the laws of logic, if the antecedent of an implication is false, the implication itself is always true, whatever the truth value of its consequent. Hence, if we assume (iii), then (i) and (ii) — that prevent any *ad hoc* solution — imply that the truth of our second statement does not depend on its content, but only on its syntax. Consequently, we shall say that it is vacuously true. Indeed (a) and (b) are quantified statements in which there is no occurrence of individual constants supposed to denote mathematical objects, such as '2', 'the empty set' etc. If (iii) is assumed along with (i) and (ii), we must conclude that all mathematical statements of the kind we have considered behave the same way. In other words, they are true only if they are vacuously so. The case may be different for statements which involve singular terms that are supposed to denote mathematical objects, like for example in '2 is a prime even number', 'All prime numbers greater than 2 are odd', '2 + 3 = 5'. Indeed, if we assume (iii), the truth of these statements, if any, depends on how we decide to treat assertions involving vacuous singular terms, such as 'Pegasus'

[20] For example, 'There is an honest politician' and 'All politicians are dishonest' are true or not depending on the existence of persons who play political roles and are or are not honest.

or ‘2’, which does not denote any object. There are several options available to deal with such statements. In any case, it should be clear that conditions (i)-(iii) imply that mathematical statements of this type can be true only based on facts concerning their syntactic form. In short, also in this case, once conditions (i)-(iii) are assumed, such statements are true only if they are vacuous, meaning precisely that their truth is based on facts that do not affect their content, but only their syntax.

Let us now consider these statements:

c) 2 is a prime number.
d) 2 is a composite number.

And again, let us assume that their syntactic structure is preserved in their reformulations into the predicate logic language. Now the problem is that several such reformulations seem possible. In particular, consider the following ones:

c') For all *x*, if *x* is 2, then *x* is a prime number.
d') For all *x*, if *x* is 2, then *x* is a composite number.
c'') There is an *x* that is 2 and that is a prime number.
d'') There is an *x* that is 2 and that is a composite number.

Just like (a) and (b), (c) and (d) seem to contradict each other, so much that, while (a) and (c) are generally accepted as consequences of the definitions previously provided, (b) and (d) are generally discarded because they are inconsistent with those definitions. Yet, nominalists who accept conditions (i)-(iii) are not only led to argue that (a) is not true, whereas (b) is true; but also that (c) and (d) are both true if they are construed in accordance with their reformulations (c') and (d'), whereas none of them is true if they are construed in accordance with their reformulations (c'') and (d''). Hence, nominalists cannot concede (unless blatantly contradicting the most basic mathematical practice) that the acceptance of mathematical statements has something to do with their truth. In other words, they are not entitled to claim that the virtues, utility and applications of mathematics depend on the truth of theorems. On the contrary, they must deny it, and explain what such virtues, utility and applications depend on.

In the fourth place and finally, nominalists could also argue that, even maintaining condition (iii), there is a sense in which we can recognize mathematical theorems as (not vacuously) true. We just need to reject at least one of the conditions (i) and (ii). Ond could hold, for example, that mathematical theorems are true within mathematics, just as the descriptions of fictional characters are true within the story in which they live. Proponents of this view, called ‘fictionalism’, deny the existence of abstract mathematical objects, but affirm that we can build mathematical stories in which these objects act, so to speak, as characters. On this view, mathematical theories are nothing but fictional stories, even though typically they are very precise and codified and built in accordance with strict rules of argumentation which allow us to distinguish between correct and incorrect statements. Accordingly, we are entitled to ascribe truth to certain statements, provided we are aware that the statements we have to do with are true within a certain story, rather than true *tout court*.

5. The Indispensability Argument

The Indispensability Argument has aroused a broad, still ongoing debate. One of the reasons why it has proven so powerful lies in the fact that its basic premises — such for example the simple observation that certain mathematical theories are constitutive of (arguably) true scientific theories — are perfectly acceptable by the advocates of nominalism. Yet, although the argument seems simple, it hides a complex set of assumptions, and among them there are also some philosophically controversial options. This is also why the most debated versions of this argument are quite different to each other, despite some essential underlying ideas. This suggests that it is of the utmost importance to spell out, even independently of the usual formulations, the essential structure of the argument and to clarify the notions involved. One point however is clear: the advantage of the Indispensability Argument (at least in some of its versions) is that, unlike many others, it paves the way for justifying the existence of mathematical objects without having to previously clarify their very nature, nor to specify the sense in which we can assume they exist. So the Indispensability Argument provides philosophers of mathematics with a tangible starting point for their analysis. Once it is established that there are objects with which mathematical theories are concerned, these theories can be investigated in order to clarify the nature of these objects and the sense in which we can talk about their existence. Of course, some could also regard this situation as indicative of a problem. In fact, how could we establish beyond doubt that certain objects exist, if we have not previously explained what their nature is and in which sense their existence is to be understood?

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A Debate on the Long Term and Reactions in Mathematics Teaching.
Is Computation Just a Technique or Can it Be a Way to Learn Mathematics?

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1. Introduction

I have chosen six images, in fact three times two very contrasted images from the same book to serve as a good introduction to what I intend to explain about how computation intervenes in mathematics teaching when studied in historical perspective. The first image is the frontispiece to Cavalieri's *Trigonometry*, as it was published in 1643 in Latin in Bologna, where we see men using devices (quadrants) that we may consider as being of a geometric kind, in order to measure country palaces and fields as well (Fig. 1). In fact, as it had very recently been invented for images in books on geometry, dotted lines are drawn in this landscape in order to explain the function of the quadrants, so distinguishing the real world from the mathematical one. Nothing however is shown concerning the content of the book itself, which is a new theory for these measurements through sinuses. This theory is exhibited through very long numerical tables, with no formulas and no geometrical figures (Fig. 1). Is the image introducing the book just a way to give some practical realism preparing the acceptance of the "hard" abstraction of the tables? One sees as well astronomical drawings of a geometrical kind on the left panel of the open door in this frontispiece. Should I add that they also are realistic drawings, preparing the acceptance of the famous astronomical computations, now considerably helped by numerical tables inside the book? We must however pay attention to the historical fact that at this time, more than ten years after Galileo's condemnation in Rome, any realistic representation of mathematics in

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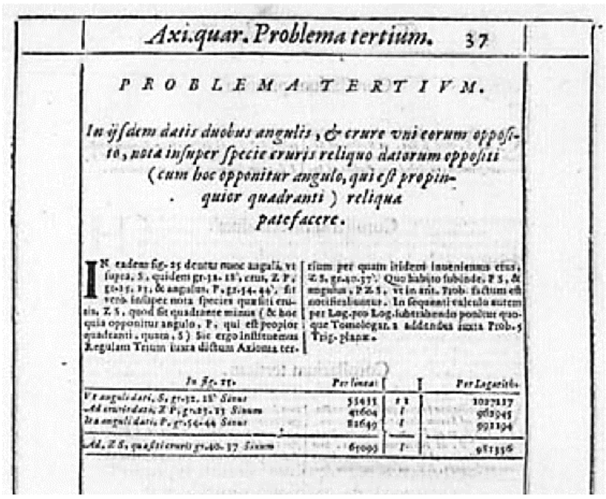
the heavens, even a geometrical one, might be considered as raising some epistemological difficulties. This may justify the dotted lines of the first frontispiece, which obviously do not exist in the real world, but sustain the idea of fictitious hypotheses in mathematics. We have to note that oppositions between at least three representations of the solar system, from Ptolemy, Copernic and Tycho Brahe, rarely made use of geometric figures. Nevertheless the numerical data were almost never mentioned. So I have here to recall that the arrival of logarithms and precise numerical tables was a novelty, if not a revolution. It was however ignored for the elementary teaching in the Universities, and in the first frontispiece we have seen.



Axi.quar. Problema tertium. 37

PROBLEMA TERTIVM.

In iisdem datis duobus angulis, & crure vni eorum opposito, nota insuper specie cruris reliquo datorum oppositi (cum hoc opponitur angulo, qui est propinquior quadranti) reliqua patefacere.

Figure 1. Frontispiece to the *Trigonometria* from Bonaventura Cavalieri, published in Bologna in 1643. In the same book, a page without any figure providing trigonomeric computations and followed by many pages of pure numerical tables.

Another contrast between symbolic writings and geometric figures may be measured with another image taken from a thesis held in 1699 in Toulon and under the supervision of a Jesuit college. The purpose of the thesis was to introduce algebra to students, and even the frontispiece pretends this algebra to be useful for naval architecture and for artillery as well. But no numerical data are exhibited, and if an algebraic formula appears (perhaps for the first time on an image), it is duly attached to a geometric environment of drawings. Even the parabolic trajectories are drawn, in dotted lines, and their representations respect the middle point property so important to Galileo's experimental and realist verification of his theory on the fall of bodies. Namely that when referring to a diameter a point on a parabola, the middle point of the subtangent segment on the diameter is the point where the diameter intersects the parabola. This can be verified on the thesis frontispiece, thus implying a knowledge of what a diameter is, and not only an axis, which was certainly not common knowledge, not so often explained even in the most advanced universities of the time, but will be part of the new advanced perspective theory to come. It certainly was not taught in the college where the thesis was held. Now in the book itself there are no geometric drawings and just algebraic items (Fig. 2b). These algebraic propositions are given without any written formula. By themselves, such formulas would have justified a serious change in the practice of mathematics teaching. So that one may say that the presence of geometry was still strong for that kind of trigonometry, but not algebra. Algebra had been introduced however in the famous *Trigonometria Britannica* published in Gouda in 1633 with excellent numerical tables, using Vieta's formula for the expression of *sin kx* in polynomial terms of *sin x*. Nothing in the frontispiece of the thesis in 1699 introduces to numerical tables. But one clearly sees mathematics being used for technical professions, as in the case of the Cavalieri's frontispiece, and we even see workers around the ship in construction. Such workers seriously contrast with the unrealistic putti that are supposed to represent young students in the Jesuit college. Workers here bring more the idea of a geometric conception for the architecture of a ship than a numerical design. The numerical aspect will come almost fifty years later, once differential and integral calculus will have found its place in naval architecture due to Euler and Bouguer, and almost never in schools. A last couple of images is taken from the frontispiece of Euler's Analysis, more specifically the *Introductio in Analysis infinitorum*, a book published in Lausanne in 1748, which was to change the practice in mathematical research. But it does not seem to have been in use in schools before it became obsolete in the early decades of the 19th century, when Analysis was rigorously organized by Cauchy as a new world with continuous functions, limits and series, as we know them nowadays. In the frontispiece of Euler's book, a table of sinus is mentioned by a title for a book on the ground, and is accompanying books on differential and integral calculus (Fig. 3). Usual tools for Euclidean geometry are seen as well, providing the impression of continuity in mathematics from the long past and of progress as well. However the setting is completely different for what is supposed to represent teaching from what we have seen in the Jesuit thesis. With a landscape seen in the background, a putto is indeed present, but with a completely changed role due to two feminine personifications. One represents a pupil learning mathematics and has to write, so certainly using formulas and numerical tables from the books on the ground, and the other is the allegory of mathematical theory that has not only to be understood but also to be meditated by the pupil. This last personification, perhaps meaning mathematical imagination as well, is then accompanied by a board on which there are geometric figures. The content of the book is in real contrast with this pleasant frontispiece: without geometric figures, it exhibits analytical formulas using the complex exponential function, from which one reduces sinus to series, and it induces a serious change for numerical tables. They can be verified at once by anybody when one needs a specific result, without having to do the enormous computations Napier had previously to conduct during so many years (Fig. 3b). Numerical analysis was no longer apart from mathematics.

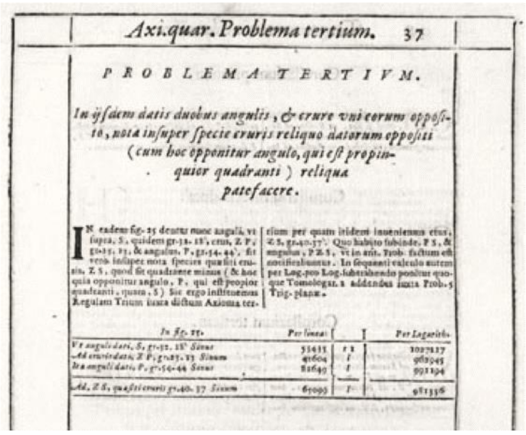


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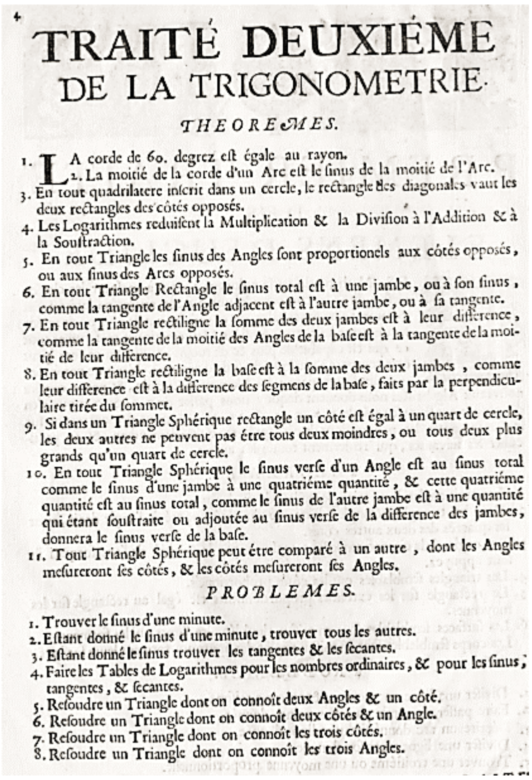
Figure 1. Frontispiece to the *Trigonometria* from Bonaventura Cavalieri, published in Bologna in 1643. In the same book, a page without any figure providing trigonomeric computations and followed by many pages of pure numerical tables.

A page in this book (page 98) with the complex exponential function and its use to construct trigonometric functions and what is now called de Moivre's formula and Euler's formula, with:

$$e^{cx} = \cos x + i\ \sin x\,, \text{with } c = \sqrt{-1}$$

Figure 2. Frontispiece for a thesis in mathematics held in Toulon in 1699. A typical page taken from this thesis, with no geometric figure and no formula as well, just problems implying computations of tables of sinus (the first item in the Problems section consists in computing the sinus of an angle of 1', but the text does not say the precision of the approximation required, and how many relevant decimal positions).



4

TRAITÉ DEUXIÉME DE LA TRIGONOMETRIE.

THEOREMES.

1. LA corde de 60. degrez eſt égale au rayon.
2. La moitié de la corde d'un Arc eſt le ſinus de la moitié de l'Arc.
3. En tout quadrilatere inſcrit dans un cercle, le rectangle des diagonales vaut les deux rectangles des côtés opposés.
4. Les Logarithmes reduiſent la Multiplication & la Diviſion à l'Addition & à la Souſtraction.
5. En tout Triangle les ſinus des Angles ſont proportionels aux côtés opposés, ou aux ſinus des Arcs opposés.
6. En tout Triangle Rectangle le ſinus total eſt à une jambe, ou à ſon ſinus, comme la tangente de l'Angle adjacent eſt à l'autre jambe, ou à ſa tangente.
7. En tout Triangle rectiligne la ſomme des deux jambes eſt à leur difference, comme la tangente de la moitié des Angles de la baſe eſt à la tangente de la moitié de leur difference.
8. En tout Triangle rectiligne la baſe eſt à la ſomme des deux jambes, comme leur difference eſt à la difference des ſegmens de la baſe, faits par la perpendiculaire tirée du ſommet.
9. Si dans un Triangle Sphérique rectangle un côté eſt égal à un quart de cercle, les deux autres ne peuvent pas étre tous deux moindres, ou tous deux plus grands qu'un quart de cercle.
10. En tout Triangle Sphérique le ſinus verſe d'un Angle eſt au ſinus total comme le ſinus d'une jambe à une quatriéme quantité, & cette quatriéme quantité eſt au ſinus total, comme le ſinus de l'autre jambe eſt à une quantité qui étant ſouſtraite ou adjoutée au ſinus verſe de la difference des jambes, donnera le ſinus verſe de la baſe.
11. Tout Triangle Sphérique peut étre comparé à un autre, dont les Angles meſureront ſes côtés, & les côtés meſureront ſes Angles.

PROBLEMES.

1. Trouver le ſinus d'une minute.
2. Eſtant donné le ſinus d'une minute, trouver tous les autres.
3. Eſtant donné le ſinus trouver les tangentes & les ſecantes.
4. Faire les Tables de Logarithmes pour les nombres ordinaires, & pour les ſinus, tangentes, & ſecantes.
5. Reſoudre un Triangle dont on connoît deux Angles & un côté.
6. Reſoudre un Triangle dont on connoît deux côtés & un Angle.
7. Reſoudre un Triangle dont on connoît les trois côtés.
8. Reſoudre un Triangle dont on connoît les trois Angles.

d

Figure 3. Frontispiece for the Introduction to the Analysis of the infinites by Euler, published in Lausanne in 1748.

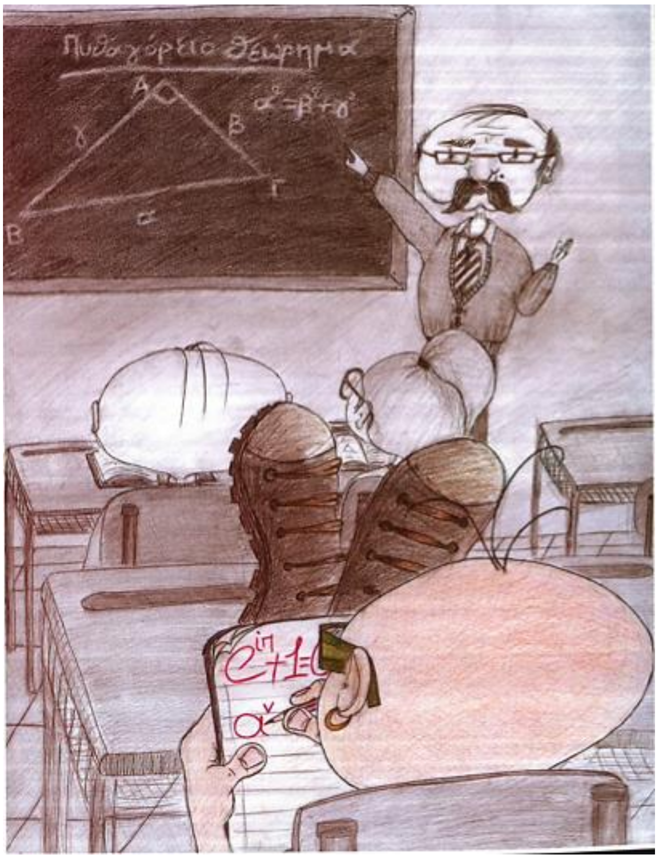


Figure 4. The opposition between a geometric figure and an analytic formula, considered as an opposition between ancients (the professor!) and modern (the gifted student).

Are the contrasts exhibited in the images of the past to advertise and to do mathematics so different nowadays? A cartoon in a calendar for 2007 produced by the Hellenistic mathematical Society (Fig. 4) is excellent for my purpose. The complex exponential function that is written by the pupil, and was introduced by Euler as already said, is still far beyond the level for secondary education in almost every country. But sinus numerical tables, which appeared in the book by Cavalieri, are everywhere used at this seconddary level of education. The cartoon provides no solution, but emphasizes a debate about the theoretical status of computation, which I think was in many respects the debate the three couples of contrasted images bore as well for the past.

One cannot think that I am just aiming at the so called backwardness of mathematics teaching to active and research mathematics, after so many years of didactic of mathematics. So many excellent specialists proved that mathematics teaching is an adaptation to a classroom and to a level expected by professors and authorities, and not just the simplification after a while of new results in mathematical research. Other studies, mainly historical ones, have also shown how research in mathematics done at a certain period of time is linked to the kind of society into which it is developed. The link is certainly not a causal one, but who can imagine that numerical trigonometry as seen in Cavalieri's book, and by the way almost fifty years after the word trigonometry was coined and tables developed to a large extent, can be completely outside the great movement towards quantification and measurement all other the world, and its circumnavigation? At least the three frontispieces show this relation of "new" numerical mathematics and world practices in the change. But this relation is forgotten in the book itself. Is it not, *mutatis mutandis*, the same nowadays? Very often we see in illustrations on a cover page of a book on mathematics some kind of advertisement to science (astronomy, fractals, Englert-Higgs boson, etc.) or to technique (computers, the first human steps on the Moon, etc.), but inside the book there is no comment upon such (perhaps good) propaganda. In the case of Fig. 2, it is clear that at the time of the image, practice of polynomial algebra was of no use to ship construction, so that my use of the word propaganda is correct.

I think that the reason of such unexplained contrasts lie at the very heart of mathematics teaching at the secondary level. In the sense that mathematics is almost a priori declared to be the best way to learn when you are young how to think in an ordered manner, what is universal and transmissible to others. But this thought, as old as Plato, does not seem to take care of the kind of mathematics being taught. And so there is a sort of automatic contradiction with the reason given to compulsory mathematics teaching, in the sense that mathematics is changing, and sometimes in a revolutionary way. No revolution perhaps for trigonometry, but certainly one with numerical logarithms and symbolic polynomial algebra, and for sure with differential and integral calculus, which were the three major changes related to the images I have shown. So there is a tendency in books for teaching to often forget about what mathematics brings to the organization of our thought by showing mathematics as used in the world that is thus transformed. It may justify why "new" mathematics, or other mathematics, are now taught. For these new mathematics, it takes time to invent not only didactical tools to acclimate the new procedures, but as well to explain why the new procedures are as helpful as the old ones in terms of organization of the thoughts of young students. This generally explains why there exist "reactions" in mathematics teaching. The word reaction, as in the political world, means that a community of teachers prefers not to adopt new mathematics, whatever they are, to come back to older practices of teaching, which are in fact reinventions. And the target of a reaction is often the numerical procedures and the computation. Not this is only one way. In the sense that we may have a reaction towards so called good old geometry against too active algebraic procedures, and this may describe the aftermath of geometry after the early 70's of New mathematics ("mathématiques modernes" as we prefer to say in French) for which logical thinking and abstract algebraic structures were the principal objects to be taught in the classroom. But in the more recent years, due mainly to the common use of electronic computers, it was also considered as good didactical matters to write papers to enhance the practice of computations and of algebraic calculus for the Secondary level. Could we not see this move towards computation as a result of a reaction against geometry this time, and so see computation as a reaction to a reaction! This way is far too simple in order to provide a complete explanation, but at least it helps to think in terms of historical tendencies in the teaching of mathematics, as we now usually think as well in terms of changes in mathematics. The question is not about Ancients and Moderns, and has little to do with the mathematics content itself, but the way this content serves the purpose of providing a place to mathematics in the general education of a mind.

My purpose here is certainly not to tell the rather long history about debates in mathematics, and images I gave are sufficient I hope to indicate some highlights of the long term story. It is not the place either to tell the details about how debates happen as well in mathematics teaching. I prefer to focus on moments of revolutions in this teaching, with the idea that they are not necessarily linked in a direct way to similar epistemological revolutions in mathematics itself. This is the major consequence of the contrasts I have shown earlier with pictures, and even I think it is interesting here to simplify by speaking of reactions, in order to take care of the generally ignored fact that the community of mathematics teachers has general tendencies and diverse movements. So this is not just for the pleasure of making history alive that I am proposing the present inquiry by using some cases only: in fact I am trying to disentangle some issues about the long term debate concerning computation in mathematics teaching by looking at the way some social, political and epistemological ingredients do intervene in the discussion of curricula.

2. Reaction against trigonometry by using geometry in the seventeenth century

For an inhabitant of a European city in the seventeenth century, outside the ordinary course of accounting practices, the most common encounter with mathematics was with sundials, and the most spectacular meridians providing noon as indicated by the name of the instrument. These dials had experienced a broad expansion since the second half of the sixteenth century. Dials gave time on which public clocks and as well watches of individuals had legally to adjust. Church bells did not chant less and in convents they respected the canonical but unequal hours. However by their construction, watches and clocks had become stable over a certain period of time, and so had led to a uniform value of the hour. The division into 24 hours all of equal length compelled in turn the division of the lines indicating hours in sundials. If the raw data of a sundial remains local according to latitude, it still depends on the season, so that the information provided in addition to a dial allowed the conventional control of the length of time of an hour and the phenomenon of the seasons. The somewhat variable speed of the diurnal motion of the Earth, and also the variable speed of its annual elliptical movement were not taken into account, even if they had become the basis of reflection of Kepler in 1609 in the *Astronomia nova*: the difference with the mean solar time may sometimes reach a quarter of an hour[1]. But who could worry about such precision? Yet there were astrologers and they pretended to establish horoscopes depending up to a second for a date of birth! This is indeed what Jean-Baptiste Morin, royal professor in Paris, claimed in his autobiography that is contained in his famous and posthumous *Astrologia Gallica* published in Paris in 1666. This extravagance in precision symbolically matched the new precision of the mathematical tables, and precisely the same Jean-Baptiste Morin had written a very good *Trigonométrie canonique* in 1657 (from a previous one in Latin in 1631), with very good numerical tables, better in fact than the Cavalieri's ones, but less precise than the Gouda's tables.

So we may think that in the schools the fact to teach how to precisely build sundials could be linked with the idea of a new exactness due to new computational mathematics. But this is not what occurred. To understand why, it is necessary first to understand what sort of a link was then felt between mathematics and sundials.

When we look for the influence of the observation of a public dial on shaping attitudes and mentalities, we are certainly led to imagine that mathematics came readily to some minds, through the finding of unequal angular deviations between the hours lines of most of the dials, a serious difference to what happens in watch dials. Nor is it yesterday as now part of common knowledge that this difference corresponds to the non-conservation of the

[1] The graph of the equation of time which oscillates about fifteen minutes, plus or minus, appeared in the nineteenth century only, and it often accompanies modern sundials.

projection angles. As essentially the hours line of a sundial are the projections of the uniform angular division of an equatorial sundial[2]. This perhaps was not known, but everyone could however know that it was not possible to carry a sundial from a city to another without having to change the hour lines, and everyone could as well question the nature of the corrections to the dial reading according to the seasons, while most people knew that the style was directed along the “world axis”, which is the rotating axis for the diurnal movement. Two mathematical aspects came readily into play: first the recognition of the importance of a geometric trace by straight lines on the dials, and secondly with the corrections from curves according to the seasons. Sundials suggested that a science of time was existing, and that it was important to master the mathematics that were thus involved. But there were two kinds of mathematics involved, if I may say so, projective geometry and numerical trigonometry. The choice in the schools was not to teach any one of those two “new” mathematics. This is precisely the story I would like to tell.

The simple form of a sundial, as the role of the shadows of the fixed style on the dial plate that places the hour lines seemed *prima facie* figures directly coming from the ordinary geometry. Such as the students could have seen in editions of *Euclid’s Elements*, which were a large part of mathematics teaching (when there was one). This impression of seniority in the image is misleading, however, as so often when it comes to geometrical figures where apparently similar figures deal in fact with quite different objectives. We know it is not easy at the sight of a single figure, to tell what proposition of Euclid can be related to it. The representation of space that governs the dials was not commonly taught and Ptolemy’s analemmas were not common: essentially, this geometry was renewed by using projections whose relative novelty was reported in a book on Optics by the Jesuit Francis de Aguilón in 1613. The focus was now on what the projections preserve: angles are preserved in the case of the stereographic projection of which de Aguilón spoke of, but projections linked to sundials were of another kind. Then came the very original procedure of Desargues, who specifically wrote a book on sundials. His invention met with no success, even if it changed deeply on the long term our conception of geometry. Yes, one may say that the singularity of three dimensional space came in addition to plane Euclidean geometric tools, so that there were now shadows in geometric books to give some concrete aspect to figures, and also dotted lines. This was in fact very different from

[2] In the equatorial sundial, hourly straight divisions are obviously equal to 15 degrees (360 /24). Thus in a vertical dial, said declining because not orthogonal to the North-South ascis, and a front facing east, the hour lines are tightened even more on the left than the declination is strong.

the idea of geometry in the Euclidean tradition where geometrical diagrams cannot exhibit objects different form the Euclidean ones- straight lines and circles[3]. Desargues brought the clever and new idea that a diagram could be seen in plane and in solid geometry as well: this gave the famous property of alignments from two triangles in perspective. But in the Jesuit colleges, which essentially represented places where mathematics was taken more seriously than in most universities of the time, the choice was generally made to carefully study Euclid's Elements as a piece of logical deduction, with no other aims than to fortify the mind on how to think. Mathematics was certainly not taught to be used for technical applications. So that trigonometry and numerical tables, considered as useful for applications, were not taught. This does not mean that sundials were not explained. But the required geometry for them had to be directly related to Euclidean propositions: something was invented, a nice didactical geometry for sundials, obviously avoiding numerical tables and trigonometry. This is what I call a reaction. Nowadays, practical books on sundials still separate the two methods, but quite often it is declared that one method is more elegant than the other (generally the geometric one), and the trigonometric one is called more precise, or more professional. The change was more drastic in a book by Bedos de Celles in the 18th century:

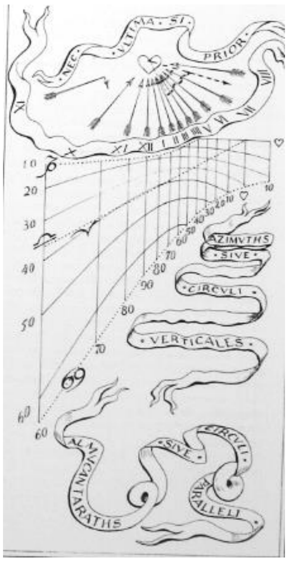
> Of these two methods, one is geometric or graphic and it operates with the simplest rule and compass. The other runs into computation. This last one is truly the best, and that of mathematics. The first is for those who do not or cannot get into the second. I would strongly urge the reader to pay attention to the method by computation. I simplified it as much as I was able. One should certainly not be afraid at the sight of a sine table.[4]

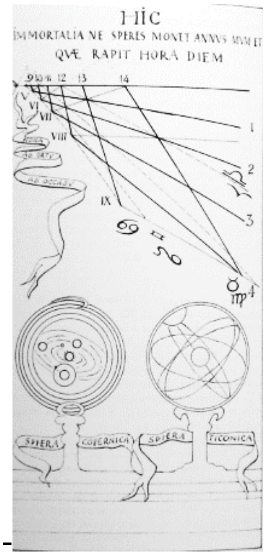
That something more than the hour was provided with sundials is remarkably shown in the year 1679, by those built for the hôtel des Ambassadeurs de Hollande, rue Vieille du Temple in Paris (Fig. 7, 8 and 9). This is the Carmelite Sebastian, in the civil Jean Truchet, who took charge of the dials, and he will soon write papers for the Academy of Sciences. Hours "Babylonian" (on the right column in Arabic numerals) and hours "italics" (in Arabic numerals in oblique) are joined by straight lines that provide information on the time in the now classical sense, with Roman writings. The scientific character, and it would have been described as mathematical during this time, is fundamentally enhanced by the display of the two drawings under the dial.

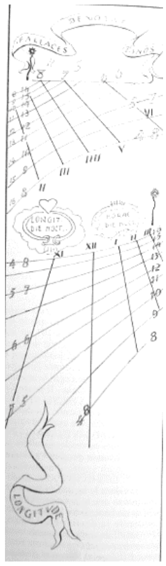
[3] See Dhombres (2003).

[4] Dom François Bedos de Celles, *La gnomique pratique ou l'art de lever les cadrans solaires avec la plus grande précision*, Paris, Delalain, 1774.

One of the drawings shows the “Copernican sphere”, that is to say the planetary system rotating around the sun, and the other the “Tychonian sphere”,







Figures 7, 8 and 9. Drawings explaining three zodiacs due to Jean Truchet on the eastern facade of the Hotel des Ambassadeurs de Hollande in Paris.

that is to say the system due to Tycho Brahe with some planets turning around the Sun, itself turning around the central Earth. Such a distinction among planets contradicted the traditional astrological interpretations. The “Ptolemaic sphere” is not forgotten, which appears in another panel, also on the eastern facade of the same hotel. As already said, such representation were rare, even in books, except for specialists and outside the college walls. We cannot forget that the ban remained from 1616 on publicly defended heliocentrism. Was the exhibit of different representations of the solar system an indirect way to show that there were still questions to be debated? If the answer is yes, we deduce that the mathematics actually involved in a dial only serve to properly present the data, and not to solve the cosmological question that fascinated so many people. This position would therefore be the triumph of Cardinal Bellarmine in Galileo’s case, who would assign a single role of fiction to mathematics in order to simplify the calculations, but nothing to do with reality. This could explain why so many Jesuit colleges wanted to keep alive the culture of sundials, but with an “old” geometry, or better said an invention in the manner of the old geometry. Note that this possible attitude nevertheless contradicted the general position of astrology, which refused to see a necessary adaptation to new systems, in order to keep geocentrism. The display in the dials respected mathematical figures that appeared in the textbooks. Such works are numerous in the sixteenth century, more or less large, and of course the larger one was in 1581 that of the Jesuit professor Clavius, over six hundred pages. The author used both the representation of space with perspective, spherical geometry, and numerical tables. This is however not the book that guided the Jesuit in their way to use dials. Because they strongly tended during the 17th century to work through geometry only, as already said. This is the main reason why I chose to speak of a reaction, without having to claim that this reaction was backwards, since we already said it is still said to be elegant. Let me consider a converse case in terms of domains, but that can be named a reaction as well.

3. Reaction in favour of algebra against geometry

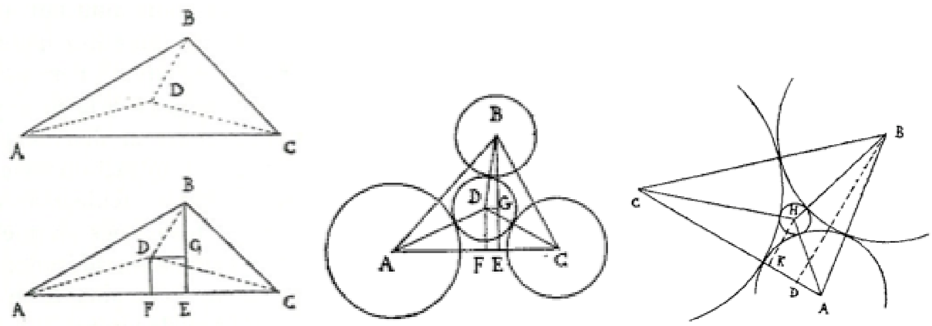
There is a painting officially showing Descartes giving “a lesson of geometry” to two seated famous ladies, the queen Christine of Suede and princess Elizabeth, wife of the then dismissed Palatinate elector (Fig. 10). Descartes is standing, as required by the Court protocol, so that he appears taller than the two aristocratic ladies.

Figure 10. An inventted representation of a proof given by Descartes to Christina, queen of Sweden, and to Princess Elizabeth of Bavaria (Louis Michel Dumesnil 1663-1739, 0,97×1,26, châteaux de Versailles et du Trianon).

The imaginary place is a room of a palace, and different groups of persons appear, all dressed in the way common for the aristocracy in the later part of the reign of Louis XIV, and years after the death of Descartes in Stockholm in 1650. The philosopher came to this city to discuss with Christine, but not specifically on mathematics. So that we may think that mathematics is here shown as a way to think. Most of these persons on the scene belong to the Court, a cardinal can easily be seen and close to him is the Prince de Condé. He is represented exhibiting the blue cordon of the Saint Esprit Order. It may be possible that the group of seven persons near the door in the middle of this fictitious representation are members of the Academy of sciences, created in 1666. It should be noted that this Academy was located in the city of Paris, and never invited to stay in Versailles. Around Descartes, it is not the university world that is represented. To the right is Marin Mersenne in the monk habit of his order (He was a Minim friar), and around him are (perhaps) two persons like Girard Desargues and Etienne Pascal, father of Blaise, accustomed to attend the weekly meetings the Minim was holding in his convent, near Place royale in Paris (now Place des Vosges), till his death in 1648.

Then the question is why the title to this painting is a "lesson of geometry". The painter did not even suggest the problem, which was possibly raised. We identify it by the correspondence exchanged between Descartes and Elizabeth in 1643, and if we know from Descartes himself that the problem was not that simple, geometry is obviously not the subject, but algebra.

> Finally I have remorse for suggesting the question of the three circles to the princess of Bohemia: for it is so difficult that not even an angel, who were equipped with instructions from Stampioen's algebra only, would be able to overcome it without miracles.[5]



Figures 11, 12, 13 and 14. Four figures drawn by Descartes for Princess Elisabeth in 1643 concerning the problem of the three circles, where he decided to take a simplified version, with three circles that are tangent two by two, like in the last

[5] *Correspondance de Descartes*, letter to Princess Elisabeth, Nov. 1643, Paris, Vrin, 1996, tome IV, pp. 26-27.

figure. Any mathematician will immediately recognize in *DG* and *DF* the coordinates of centre *D* with respect to two orthogonal axes[6].

I do not need to go further in the discussion, as four figures from Descartes' letter (Fig. 11 to 14) prove that he attempted to guide Elisabeth in the new analytic geometry he had invented, and wanted to avoid her having to tackle the problem in too difficult a way. In the *Apollonius Gallus* in 1600, Vieta had geometrically solved the problem of finding a circle tangent to three given circles, using properties of similar figures, notably that tangency between curves is preserved for similar figures. Descartes was restricting the problem to three circles, two by two tangent, but instead of a geometric construction, he asked for a quantitative measure of the radius of the looked for circle[7]. Computation gave far more than a geometric construction. I just quote the second degree equation written by Descartes for this problem, because it shows that the problem may have two solutions in terms of the radius, and this could be said almost a priori using this algebraic methods. That was the lesson for the mind Descartes wanted to teach.

$$\begin{array}{lcl} ddeeff & \infty & 2deffxx + 2deeffx \\ + ddeexx & & + 2deefxx + 2ddeffx \\ + ddffxx & & + 2ddefxx + 2ddeefx \\ + eeffxx, & & \end{array}$$

Even if it looks simple to us, by its intrinsic algebraic technique, the very problem on the table with Fig. 10, if I dare say so, could not be represented, even in the early eighteenth century when the painting was produced. There is also no trace of this problem in Jesuit colleges during the seventeenth century. Recall that Descartes' work was not accepted, having been put in the *Index librorum prohibitorum* in Rome. So that the frontispiece we have seen with the 1699 thesis (Fig. 2) may appear as a disguised quotation from Descartes, and the application to naval construction is not technical algebra, but the use of the Cartesian analytic method of thought. However, the painter of

6 (pp. 39, 40 and 48 of the edited correspondence).

7 In modern writings, if *a*, *b*, and *c* are the inverses of the radius of the three given circles, which are tangent two by two, and *r* the radius of the looked for circle, there is an algebraic relation:

$$\left(a+b+c+\frac{1}{r}\right)^2 = 2\left(a^2+b^2+c^2+\frac{1}{r^2}\right)$$

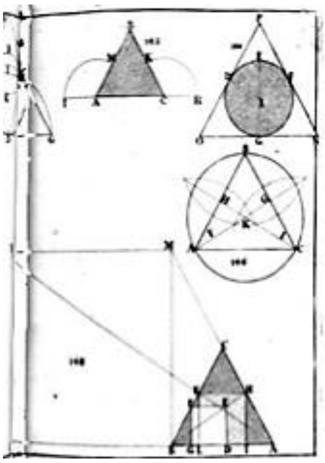
So that the second degree equation for $1/r$ can be solved providing in a rather simple way two solutions, that is two possible circles only:

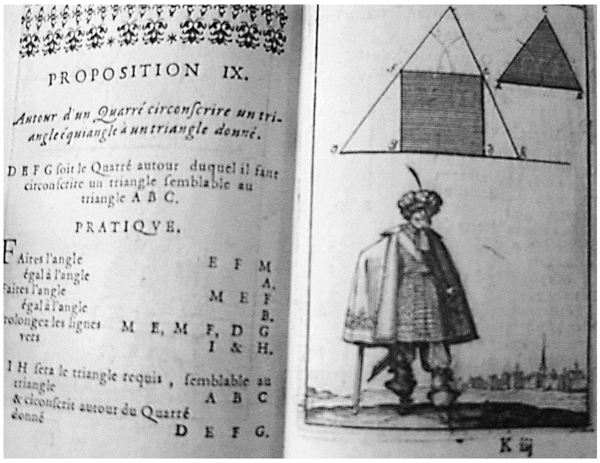
$$\frac{1}{r} = a+b+c \pm \sqrt{ab+bc+ca}$$

the scene with Descartes duly avoided exhibiting compasses, contrary to what was usually found in the iconography, as soon as mathematics was involved. Could we deduce that it was a way to signal the new analytic geometry? This was addressed to some happy few only, has nothing to do with a reaction, but was a way to promote new mathematics with the maintained idea it serves for the general education. The expected reaction to this disguise of algebra by geometry did not take place in the 17th century, but during the next century. Many mathematicians, Euler and Lagrange for example, and mathematics eachers as well, avoided to draw geometric figures in textbooks.

4. The other actuality of geometry, which does not seem to have reached classes

I should arrive at a completely different situation about what is usually said to be symbolic algebra, and I prefer to speak of polynomial algebra. This was launched by Descartes in a book precisely organized in order to teach how to



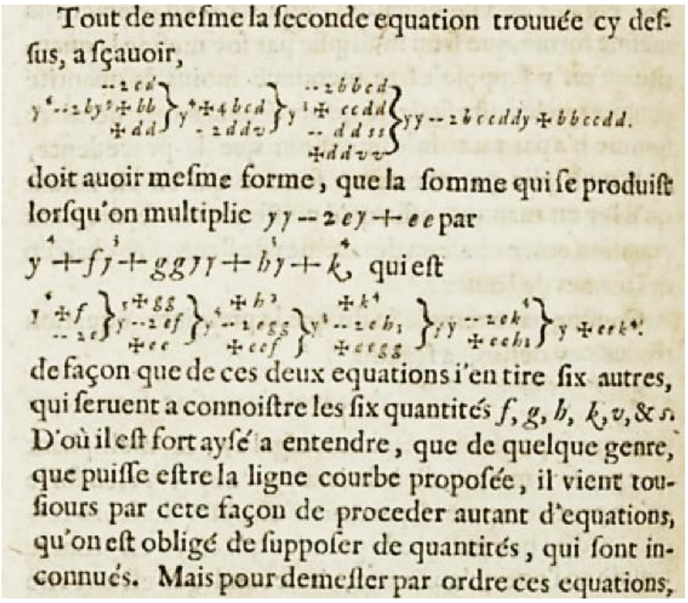


Figures 15 and 16. The construction of a square inscribed in a triangle, as explained in a plate of an edition by Albert Girard in 1651 from the *Traité et practique de Géométrie et premi-èrement de l'usage du compas* of Samuel Marolois. The construction is done in the last figure from similarity in two ways, from square inside the triangle, and another one outside. — A double page from *Pratique de la géométrie sur le papier et le terrain* de Sébastien Leclerc, published in Paris in 1682, with the problem of circumscribing a square by a triangle similar to a given one.

think, the *Discours de la méthode*. In the specific Essay devoted to mathem-atics, and curiously named Geometry while mainly algebra was discussed for the setting of what we now call"analytic geometry", the purpose was not to present a course of mathematics. But to offer new procedures of writing mathematics, and to show new ways of computing in order to think about geometrical problems. In fact the revolution in polynomial algebra was to give computations a spatial setting. Coefficients are ordered in a vertical way, powers of the variable are ordered in a horizontal way (Fig. 17). This gives a sort of a swing of the head to the one conducting the multiplication of two polynomials. But this is also the way to see that three coefficients are at most concerned when multiplying by a polynomial of degree two, as in the case of finding a double root. Therefore the linear system deduced for the method of indeterminate coefficients is always solvable. If the kind of pedagogy developed by Descartes to introduce this new way of computing is perhaps too concise, it shows at least how to think with polynomials. I have to add that it took almost a century for Descartes' way to enter the schools, and unfortunately I am not sure that the wish of Descartes about the help of computation for the mind was respected in the classes, even today. But I must add that his wish was satisfied in the classroom with analytic geometry during the 19th and 20th century. Is there not a reaction nowadays? I leave this to the reader to think about it.

5. Conclusions

Of course, the historian has a considerable advantage due to the hindsight and therefore of the knowledge of what happened long after the events he evokes. His task is not to recreate events the way all contemporaries have experienced them, being ignorant about the future, but his work, less a fiction than a critical discussion, is to identify trends that can be seen as preparing a following, as well as other trends that were not pursued. In a still unexplored domain that could be called the historical didactics of mathematics, the advantage is to demonstrate the presence of contradictory forces in mathematics education. I chose here to describe reactions. Not because they are the only ones resulting from what we may generally call debates, but because to look at such reactions helps to understand how teaching mathematics imply choices. These choices are not the result of an irrational behaviour, and are not only related to the delay between teaching and research. These choices have obvious connections with mathematics in the making, and this is where the word reaction



Tout de mesme la seconde equation trouuée cy dessus, a sçauoir,

$$y^4 - 2by^3 \left.\begin{matrix}-2cd\\+bb\\+dd\end{matrix}\right\} yy \left.\begin{matrix}+4bcd\\-2ddv\end{matrix}\right\} y \left.\begin{matrix}-2bbcd\\+ccdd\\-ddss\\+ddvv\end{matrix}\right\} yy - 2bccddy + bbccdd.$$

doit auoir mesme forme, que la somme qui se produist lorsqu'on multiplie $yy - 2ey + ee$ par

$$y^4 + fy^3 + ggyy + h^3y + k^4,$$ qui est

$$y^6 \left.\begin{matrix}+f\\-2e\end{matrix}\right\} y^5 \left.\begin{matrix}+gg\\-2ef\\+ee\end{matrix}\right\} y^4 \left.\begin{matrix}+h^3\\-2egg\\+eef\end{matrix}\right\} y^3 \left.\begin{matrix}+k^4\\-2eh^3\\+eegg\end{matrix}\right\} yy \left.\begin{matrix}-2ek^4\\+eeh^3\end{matrix}\right\} y + eek^4.$$

de façon que de ces deux equations i'en tire six autres, qui seruent a connoistre les six quantités $f, g, h, k, v,$ & s. D'où il est fort aysé a entendre, que de quelque genre, que puisse estre la ligne courbe proposée, il vient tousiours par cete façon de proceder autant d'equations, qu'on est obligé de supposer de quantités, qui sont inconnues. Mais pour demesler par ordre ces equations,

Figure 17. An extract8 of the *Geometry* by Descartes published in 1637 in Leyden, where one sees the first writing of a general polynomial of degree four, the vertical disposition of coefficients in a product, and the process for the method of indeterminate coefficients.

is best validated, but there are connexions too with both social and epistemological crises affecting societies. Such was the crisis I mentioned in this text following Galileo's condemnation. In conclusion, I would like to raise a question in form of a snub to PISA. What could indeed have given tests in 1650, if one had tried to identify performances in computations in analytic geometry ? After all, the age of investigation today, age 15, corresponds to the average age of students in their first year of college long ago. If PISA scientifically examines some performances, is it not that inevitable choices, like those I mentioned for the 17th century, are hidden by the so-called international guarantee of neutrality?

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[8] See, Descartes (1637: 350).

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Mathematics and Education: Endeavors for Survival

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The main threats to sustained human existence now come from people, not from nature. Ecological shocks that irreversibly degrade the biosphere could be triggered by the unsustainable demands of a growing world population. Fast-spreading pandemics would cause havoc in the megacities of the developing world. And political tensions will probably stem from scarcity of resources, aggravated by climate change. Equally worrying are the imponderable downsides of powerful new cyber-, bio-, and nanotechnologies. Indeed, we're entering an era when a few individuals could, via error or terror, trigger societal breakdown.

M. Rees, *Science* editorial, v. 339, 08 Mar 2013.

Floods and fire and convulsions and ice-arrest
intervene between the great glamorous
civilizations of mankind. But nothing will ever
quench humanity and the human potentiality
to evolve something magnificent out of a renewed chaos.

D.H. Lawrence, *Fantasia of the Unconscious*, 1922.

1. Introduction

Education is a practice present in every culturally identified group. The major aims of education are to convey to new generations the shared knowledge and behavior and supporting values of the group, and, at the same time, to stimulate and favor progress.

Ubiratan D'Ambrosio, "Mathematics and Education: Endeavors for Survival"
in Laura Branchetti, *Teaching and Learning Mathematics.*
Some Past and Current Approaches to Mathematics Education, pp. 77-95.

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http://isonomia.uniurb.it/epistemologica

Let us consider groups of individuals who share modes and styles of knowledge and behavior, supported by a system of values, which were generated and accumulated throughout a common past. This characterizes a culture. Thus, a culturally identified group, be it a professional guild, a family, a community, a nation, share sets of modes and styles of knowledge and behavior and values, ingrained in traditions, which support knowledge and behavior. Knowledge, behavior and values which come from the past, justify present behavior and, at the same time, entice and make it possible the advancement of knowledge. Inevitably, the supporting values also go through permanent revision. This is the essence of progress.

The phenomenon of globalization leads us to consider a much larger group, indeed the total group of the human kind. This leads to envisage a universal culture. The major challenge is to recognize shared knowledge and behavior and supporting values for this total group, that is, for the human kind. This asks for universal and transcultural knowledge, behavior and values. Examples of transcultural and universal knowledge are mathematics and the sciences in general. Modern behavior, euphemistically called civilized behavior, is mainly expressed in the appropriation of technology and the dominance of economy. Particularly the media, which is advancing worldwide as an universal behavior. A strong piece of resistance is, as it has always been, throughout history, the systems of values.

Education has been focusing on knowledge, behavior and values of culturally identified groups and on past struggles for keeping the identity of the group. The violent facet of the struggles has dominated the historical narratives within education. If we accept the initial premises that action in the present reflects the past, it is undeniable that education has been favoring violence. The historical narratives are impregnated with hostilities and atrocities, and emphasize moments of success or failure. Although the moments of temporary success are, sometimes, marked by efforts to build up new styles and modes of knowing, behaving and accepting different values, these efforts have not been deserving significant attention in history education.

Every human being experiences biological, physical, social, psychological, spiritual needs and also wants. A road to peace is to achieve a balance between needs and wants. Education for peace must consider the realms of inner peace, social peace and environmental peace, thus paving the way to military peace. These four dimensions of peace are intimately related. To achieve peace between human beings, we must understand how man is integrated in nature and we must respect the equilibrium that exists in nature. This means, to be in peace with the environment. Taking advantage of natural resources to accumulate wealth to a few, which is perpetrated at a structural level of the economy, generates social injustices which cause violence.

In this paper I will discuss Mathematics, the earliest and most recognized universal system of knowledge, which originated in the Mediterranean Antiquity. As it has been said by historian Mary Lefkowitz: "The evolution of general mathematical theories from those basics [mathematics of Egyptians, Sumerians and others] is the real *basis of Western thought*"[1] (italics are mine).

History shows that mathematical ideas, essentially explicit or implicit strategies of observing, comparing, classifying, ordering, measuring, quantifying, inferring, are the bases of the Arts, the Religions, the Sciences and, in modern civilization, the technological, industrial, military, economic and political complexes, are in the origin of Mathematics. It is important to recognize how Mathematics and mathematicians have benefited from the Arts, the Religions, the Sciences and the technological, industrial, military, economic and political complexes and continues to draw resources from them for their continuing progress.

This paper deals, basically, with the global responsibility and the commitments of mathematics educators to mankind.

I will start by looking at the objectives and goals of mathematics education. In other words, putting the question: why do we teach mathematics in schools? This is a recurring question, basic for any discussion of mathematics education[2]. But we might even propose a more general question: why do we have schools? One reason is that it is for the transmission and generation of traditional and also of new ways, modes, styles and techniques of explaining and coping with the natural and cultural environment. The totality of ways, modes, styles and techniques of explaining, of understanding and of coping with the natural and cultural environment, vary from one cultural and natural context to another. This is what I call the Program Ethnomathematics[3].

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[1] Interview given to Ken Ringle, *The Washington Post*, June, 11th 1996.

[2] In ICME3, Karlsruhe, 1976, I prepared a working document on "Overall Goals and Objectives of Mathematics Education" for the Work Group on "Why Teach Mathematics?". A summary of the working document was published in Athen & Kunle (1979: 190-209).

[3] An overall view of the concept of Ethnomathematics, see D'Ambrosio (2002).

There is a growing interest, worldwide, in Ethnomathematics[4]. Not only on the historical and epistemological approaches offered by Ethnomathematics, but mainly because of the pedagogical implications of this approach. Later on this paper, the Program Ethnomathematics will be further discussed.

No one denies that quantitative thinking dominates in the workplace, in economy, in education, in health care, and in just about every sector of modern society. Modern science, hence modern thought, is supported by quantitative reasoning. Greek philosophy, upon which later Mediterranean civilizations built their systems of thought, were supported by qualitative thinking. The same is true with every other system of explanations. Ethnomathematics privileges qualitative reasoning. But the emergence of modern science, which can be traced back to the success of Isaac Newton's *Principia,* determined the dominance of quantitative reasoning.

Modern civilization is quantitative. Social and political organization is quantitative. Literacy is incomplete if limited to reading and writing. To be quantitative literate is essential for a democratic society, as explicitly stated in a *Report of The National Council on Education and the Disciplines*, of the USA: "If individuals lack the ability to think numerically they cannot participate fully in civic life, thereby bringing into question the very basis of government of, by, and for the people."[5]

There is a growing interest in Ethnomathematics among educational activists. It was not a surprise to me when I was invited, a few years ago, to give a lecture with the title "Is Ethnomathematics Revisionism?"[6]. The question was, obviously, politically targeted. Indeed, revisionism came to the dictionary of ideas in the context of critical thinking about Marxism, and became a label to denounce conflicting political and ideological positions. Soon it was adopted in historiography. Since all the fields of knowledge, the political and social sciences, the religions, the arts and the sciences have always developed in a symbiotic relation, it is not surprising to ask if Ethnomathematics is revisionism. The answer cannot be a simple yes or no.

In March 1994 I was invited to give a plenary talk on Ethnomathematics in the annual meeting HIMED 94, organized by the British Society for History of Mathematics. A few days before, London University had issued a diploma's course guide which included, among other new subjects: Political

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[4] The *International Study Group of Ethnomathematics/ISGEm* was founded in 1985. It publishes the *Newsletter of the ISGEm* twice a year. On-line: http://www.rpi.edu/~eglash/isgem.htm.

[5] Steen (2001).

[6] D'Ambrosio (1996).

roles for mathematics education; Motivation, affect and inequalities; Mathematics and gender; *Ethnomathematics*. Coincidentally, in the eve of my talk, *The Observer* (London, March 27th, 1994) published an editorial entitled *Education's guerillas prepare for war*, where we could read the following comment:

> Ethnomaths? This is the maths we pick up by chance in day-to-day life, said to be as valid, if not more so, than the maths we're taught in school. So it follows, as the guide helpfully explains, that classroom teaching merely confuses and demoralizes the pupil. Education is thus reduced to no more than the serendipity of random experience.

This misleading statement is an evidence of a defensive posture against the challenge to the prevailing system of knowledge.

Some 25 centuries ago, the Greek philosopher Hippocrates proposed an oath which has been adopted since then by the medical professionals[7]. Even today, candidates to admission to the medical profession, sworn the Hippocratic Oath. Although both Education and Mathematics exist in every culture since immemorial times, even before Hippocrates, we do not have a similar oath.

Essentially, we have not been explicit in our practices about a comprehensive ethics, which synthesizes our commitment and our responsibility. Our commitment to children is to convey a broad view of the world and of mankind in general. Our responsibility is to infuse a broad and general ethics of respect, solidarity and cooperation. This means a broader concept of Social Justice. There has been much emphasis among Mathematics Educators in Social Justice as opportunities of access for underprivileged classes. No doubt this is important, but even more important is to offer to the entire mankind a sustainable life. This is strongly threatened. I discuss this in a recent paper[8].

It is important to regard children as individuals with hopes and expectations of a future which reflect their own individual and cultural history. We can hardly exert our profession without a reflection about the future and a perception of the state of the world and a serious concern about mankind as a whole. These are essential to help the children to acquire a consciousness of their position in the world.

The following questions are thus essential for educators: what do we know about the children we are teaching and their cultural background? What

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[7] It is interesting that most of the arguments leading to the oath refer to the relations between teachers and apprentices. Apprenticeship seems to have been discarded in modern education. The importance of a "role model" seems to be ignored in educational theories.

[8] D'Ambrosio (2012: 201-213).

do we know about the future? What is the state of the world? What does it mean “mankind”? What is our role, as teachers, in influencing the future?

In a way or another, these questions have partially guided the preparation of teachers. Psychology, Learning, Curriculum Development, Sociology and other topics became part of teacher training programs as a response to these quests. Regrettably, usually there is no explicit mention of the general objectives which justify the presence of these subjects in the curriculum. All these disciplines, particularly in programs of Mathematics Education, have a propaedeutic character, with much emphasis given to contents and methodology and less attention to overall objectives.

Indeed, disciplines are part of the structure we are studying and they represent the object of our analyses — knowledge — rather than the tools. It is very difficult to avoid the traditional approach which is focused on disciplines, since they are part of our intellectual history. But we have to escape the boundaries imposed by the disciplines and by centrist visions of history. We are particularly interested in Mathematics and Education. Mathematics has taken different styles and different forms in different times and in different natural and cultural environments[9]. The same happens with Education.

2. The state of the world

The state of the world is disturbing. It is becoming common to express the discontentment, in fact the disenchantment, with the course civilization is taking, by chastising Science and Technology, which are recognized as the embodiment of modern society. Science and Technology are thus blamed for the malaise of humanity. Mathematics is obviously directly affected by these trends.

Criticism is faced with a mounting reaction. It is difficult to deny that Modern Science tries to organize facts under principles and laws and that Western thought has been focusing on experiences which under specifiable conditions are available to everyone, thus open to different interpretations. This has been regarded as a danger of opening doors to anti-science and pseudo-science. A defensive attitude almost invariably follows.

A similar argument is sometimes used in referring to contextualized learning, particularly to Ethnomathematics.

[9] The historian Oswald Spengler (1932: 52) says, in his seminal book *The Decline of the West*: “there being consequently as many mathematics — as many number-worlds — as there are higher Cultures.

To challenge knowledge, both scientific, religious, socio-political and historical, does not mean retrogress. It has always been a coherent response to the state of society and it can be understood if we look into the full cycle of knowledge in a historical perspective, of course freeing ourselves of epistemological biases which are adopted to justify the prevailing socio-political and economical order.

The essence of these biases is the argument that Science is an object of knowledge of a different nature, in the realm of the *ratioïd*. This neologism was introduced by Robert Musil, to signify different, but exclusive, ways of understanding and explaining facts and phenomena of reality. This is particularly strong when we conceptualize the Program Ethnomathematics.

Knowledge is generated by individuals and by groups, and it is intellectually and socially organized and diffused. This full cycle of generation, organization and diffusion of knowledge intertwines needs, myths, metaphors, and interests. The human species develop, like other animal species, strategies of hierarchical power. Control of knowledge is intrinsic to hierarchical power[10]. The usual concept of "empowerment" ascending the ladder of hierarchical power, is a most subtle instrument of domination. To share knowledge is essential for the survival of the group (family, tribe, society). But the sharing is limited, through the use of filters, with the objectives of maintaining the prevailing hierarchical power[11]. These filters have as a consequence the denial of access to the full cycle of knowledge. This is a profound philosophical question, clearly illustrated in the first explicit prohibition in the monotheistic tradition: "You may eat indeed of all the trees in the garden. Nevertheless of the tree of the knowledge of good and evil you are not to eat, for on the day you eat of it you shall most surely die"[12]. The trap and the moral issues involved with empowerment is well illustrated in the classic movie *Blade Runner,* of 1982, directed by Ridley Scott.

The matter is essentially political. There has been reluctance among mathematicians, to some extent among scientists in general, to recognize the symbiotic development of mathematical ideas and models of society. Is in amply recognized that Mathematics has grown parallel to the elaboration of what we call Modern Civilization.

In reflecting about the current state of the World, it is not so important a question of priority. To agree with Lefkowitz, cited above, when she says that the Egyptians, the Sumerian and other civilizations were ahead of the Greeks,

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[10] A good discussion of power related to knowledge, in particular mathematical knowledge, can be found in Sal Restivo (1991).

[11] This is discussed in D'Ambrosio (1992: 205-214).

[12] On-line: http://www.unz.org/Pub/Bible-1966v01-00015?View=PDF

but that the contribution to build up general mathematical theories was indisputably Greek, or to recognize that the medieval scholars received Euclid through the Arabs, is irrelevant. The important fact is that the discipline Mathematics, as it is recognized today in the academia, developed parallel to Western thought (philosophical, religious, political, economical, artistic, cultural). To look for Mathematics in all aspects of Western Civilization is redundant, because, undeniably, Mathematics and Western Civilization belong to each other.

In my conference in the XIII CIAEM/Iberoamerican Conference in Mathematics Education, held in Recife, Brazil in 2011, I suggested interpreting the essence of Mathematics as, metaphorically, the Roman mythological god Janus, with two faces, looking to the future and to the past. I see mathematics with two faces, one the support of developing armaments of mass destruction and an unequal instrument for the accumulation of resources by the few members of the power elite, typically for the support of ferocious capitalism, and the other as an intellectual instrument to explain and demystify facts and phenomena, to allow for the generation and development of goods so important for the well being of human beings. I have discussed this in a text on the possibility of a "NonKilling Mathematics"[13].

3. Questioning systems of knowledge

When we question the current social, economical and political order, we are essentially questioning the real threat to its continuation of Western Civilization. How is it possible to avoid questioning its pillars, Mathematics and Science? We can not disregard, in this questioning, those considered, by the criteria of academia, non-mathematicians and non-scientists. Many of the non-mathematicians and non-scientists have much to say about major issues threatening the survival of humanity. The resource to arguments of academic competence leads to intimidating language and to passionate arguments. How can we reach the new by refusing, discouraging, rejecting, denying the new? Indeed, a subtle instrument of denial is discouragement through intimidation. Language plays an important role is this process, as every school teacher knows.

Particularly in Mathematics, the use of a formal language inherent to academic Mathematics has been a major instrument for determent of new

[13] Ubiratan D'Ambrosio (2009: 239-268). On-line: http://en.wikiversity.org/wiki/Nonkilling Mathematics

thought. It is academic mobbing, a form of bullying. Research done by Sociologists is growing on the studies of the relations of Science and Society. But the new field of Social Studies of Science has been chastised. Alan Sokal draw much attention to the theme in a hoax published in one of the cherished journal of the postmodern critics[14]. This started the polemic which became known as the "Science War", not different from those polemics focusing afrocentrism and feminism. The polemic reveals that the issue is a political one.

The language of mathematics and science, the same as norms and epistemologies, differ from group to group, from society to society, and are incorporated in what is called culture. The crux is the dynamical process of the encounters of cultures and the resulting mutual expositions, which underlie the construction and reconstruction of knowledge and the maintenance, substitution, dissolution and modification of epistemologies and norms. When this process is dominated by authority, as it was the case in the colonial process and indeed characterizes conservative schools, the outcome is unavoidably to limit and even to denial the full participation of society in this dynamical process[15].

Social and political scientist Marcus G. Raskin and physicist Herbert J. Bernstein, in their analysis of the linkage between knowledge generation and political directions, claim that:

> Science seeks power, separating any specific explanation of natural and social phenomena from meaning without acknowledging human attributes (such as love, happiness, despair, or hatred), the scientific and technological enterprise will cause profound and debilitating human problems. It will mask more than it tells us about the universe and ourselves.[16]

The criticism inherent in reestablishing the lost connection of the sciences, technology and human values is causing unavoidable conflicts. This is no less true about Mathematics, in which the acknowledgement of human attributes is conspicuously absent in its discourse.

This has not been so in the course of the History. Mathematics, the same as in the History of Sciences, which can not disregard their trans-disciplinary

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[14] See the polemics around the article by Alan Sokal (1996) published in Social Text, chastizing posmodernism, particularly Sociologists of Science, and also the article by Weinberg (1996: 11-15). Particularly interesting are articles by Sullivan (1996: 1127-1131), and by Harrell II (1996: 1132-1136). The exchange of letters between Noam Chomsky and Marcus G. Raskin, printed as Chapter 4 in the book in note 9, pp. 104-156, is also very illustrative.
[15] A caricature of this denial is in the story of Baum (1900).
[16] Raskin & Bernstein (1987: 78).

characteristic, bringing into broad consideration religious, political, economyical, social and cultural issues. Regrettably, current Epistemology and History, and above all the educational framework, are constructed to justify the prevailing socio-political and economical order, in which we recognize different "theories of science". The philosopher of science Imre Lakatos proposed, in a seminal work[17], three main currents:

> *Skeptical*: represented mainly by Paul Feyerabend, claiming that any system of knowledge is as good -- or as bad -- as any other.
>
> *Demarcationism*: represented mainly by Popper and the same Lakatos, which essentially distinguishes between good science and bad science and while they recognize that scientific results are mutable, the methodology is, like religion, doctrinal.
>
> *Elitism*: represented mainly by Kuhn and Polányi, which essentially claims that only scientists can distinguish and establish criteria for telling good science from bad science.

Raskin and Bernstein add[18] a fourth explanatory direction: "*Reconstructionism*: which views science as an humanistic activity and looks for its roots in faith and political power."

Mathematics usually finds its explanation in Demarcationism and Elitism, in which the world of ideas prevail. I see in all these directions a limitation, lessened but still present in Reconstructionism. I propose another explanation:

> *Holism*: which sees the generation of knowledge as the result of a complexity of sensorial, intuitive, emotional, mythical and rational factors. We are "informed" by these factors and process the information in a way as yet unknown in the stage of knowledge of how the human mind functions. This holistic approach to knowledge owes much to artificial intelligence, biology and sociobiology.[19]

Although it has been common to place Mathematics in Demarcationism and Elitism, the growing movement of Humanistic Mathematics gives new breadth to this discussions and places mathematical knowledge in the holistic approach[20].

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[17] Lakatos (1978: 107).
[18] *Ibidem,* note 14.
[19] D'Ambrosio (1981: 33-42). I am particularly indebted to Wiener (1948); Maturana & Varela (1987); Lumsden & Wilson (1981).
[20] The Humanistic Mathematics movement started in 1986, under a personal effort of Alvin M. White. It publishes the *Humanistic Mathematics Network Journal.* On-line:

Let us examine the question of political power. A recent study in the USA claims that only 35% of students said they spent six or more weeks studying or doing homework during senior year in high school and that 33.9% students report being bored in class. There is no point in claiming that about half of the current generation is intellectually "lost". There seems to be a perverse consensus that the loose end is in the young side. Maybe some attention should be given to the other end of the rope. The problem does not reside in youth, but in school, even more in society. As Fred M. Hechinger puts it:

> The drift toward a society that offers *too much to the favored few and too little to the many* inevitably raises question among young people about the rewards of hard work and integrity (italics are mine).[21]

Thus we agree that the real problem facing education is political, essentially the result of an unequal distribution of material and cultural goods, intrinsic to modern economy. This is clearly seen when we have a critical look into the concepts of property, production and other global issues in modern society[22].

Some readers will claim that this has not much to do with Mathematics and less with Mathematics Education. I claim these issues have everything to do. Education is a political endeavor, and the political dimensions of Mathematics and Mathematics Education are impossible to deny[23]. The cultural consumerism practiced in schools and in academia, which manifest in testing as quality control, have been efficient in trimming processes and focusing only in results. The discussions on processes, which is the crux in challenging the prevailing systems of knowledge, have been practically inexistent in Mathematics Education. The Program Ethnomathematics focuses particularly on this remark[24].

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http://www3.hmc.edu/~jnelson

[21] Hechinger (1992: 206).

[22] See, for example D'Ambrosio (1998: 453-461); The book by Margalit (1996), is quite challenging. The International Network of Scientists and Engineers for Social Responsibility/INES offers a good electronic forum for discussion of these basic issues, on-line: http://www.inesglobal.com

[23] Three conferences of the group PDME/Political Dimension of Mathematics Education were realized: 1995: Bergen, Norway, 1993: Cape Town, South Africa, 1990: London, UK. Proceedings of all three are available. In the Eight International Congress of Mathematics Education/ICME 8, in Seville, Spain, July 14-21, 1996, the WG 22: Mathematics, education, society, and culture, chaired by Richard Noss, focused on the political dimensions of Mathematical Education. The book of Frankenstein (1989), is representative of this movement.

[24] See the book by Powell & Frankenstein (1997).

Mathematics and its history are delivered as frozen systems of knowledge. There have been few writings about values attached to Mathematics and even less about the moral quality of our action. To search for a correlation between the current state of civilization and mathematics has been uncommon among mathematics educators[25]. Particularly the political component, which was so well studied by Paulo Freire, Michael Apple, Henry Giroux and others with respect to education in general, seem to have drawn less attention of Mathematics Educators. Even less has been done with respect to employability. An explicit reference to this important aspect of societal life was made the science educator Michael R. Matthews:

> There is a fiscal crisis of the State, and a structural unemployment crisis. Increasingly, there will be a small percentage of people whose labour time is virtually priceless and a vast percentage whose labour time is basically worthless.[26]

Indeed, we are cheating our youth insisting in the traditional curriculum. Probably the best research in this field is due to Paul Willis, who shows how the practices in schools inculcate conformist ideology and the normality of prevailing social order. Willis does not refer specifically to Mathematics[27]. French sociologist Viviane Forrester extends the criticism to the entire organization of labor[28]. The movement ATTAC/*Association pour la taxation des transactions financières pour l'aide aux citoyens* focuses on the perverse financial globalization[29].

Strangely, the educational proposals of Robert B. Reich have not drawn attention of mathematics educators[30]. It proposes a new vision of education for the information age. Recently, I proposed a new concept of curriculum, changing the priority from delivering information to providing *critically* the instruments for being operational in the modern world, which I call literacy (communicative instruments), matheracy (analytical instruments) and technocracy (material instruments)[31].

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[25] Usually these appear in the context of Mathematics Education when goals and objectives are discussed. These practically ignore, revealingly, deeper issues relating to man and society and the political role of Mathematics Education. These matters have been discussed in my paper mentioned in note 1.
[26] Matthews (1980: 197).
[27] Willis (1997).
[28] Forrester (1999).
[29] *Tout sur ATTAC*, Éditions mille et une nuits/Librairie Arthème Fayard, Paris, 2000.
[30] Reich (1992).
[31] D'Ambrosio (1999: 131-153).

4. Mathematics, rationalism and ideology

To a great extent, the polemics around the postmodern discourse of sociologists of science reflects the ideology intrinsic to words. Indeed, language has been, in the course of history, the main instrument in supporting the power structure and denying free inquiry. Academia and society in general feels threatened by language. One must be reminded that the major confrontations of the sixties started with the free speech movement. This affects particularly schools. Although considering it to be of fundamental importance, particularly for Mathematics Education, I will not get into discussions about languages issues in education, particularly bilingual education.

Language, in various forms — gesture, speech, written, drawing and other forms — is the communicative instrument for excellence of the human species. Languages synthesize the complexity of sensorial, intuitive, emotional, mythical and rational factors. It comes to my mind the case of a school teacher who asked children to draw a color picture of a tree seen through the window of a classroom. Jane came with a tree painted red. The teacher corrected the child, even suggested to the parents that Jane might have a vision problem! A few days later the teacher was sitting in the same place as Jane was, at the same time of the day, and the Sun was in the same position. The teacher saw the tree red.

Many say that this example is misleading, since it does not deal with objective reason. Indeed, as math teachers we are told that we have to teach objective reason, to stimulate rational thinking in our students. But human mind is a complex of rational, emotional, intuitive, sensorial and mythical perceptions, involving all at the same time, with no hierarchy. Maybe we have been emphasizing too much the rational dimension and denying, even rejecting and repressing, the others. It is not uncommon to see a child punished for being "too happy" in the classroom. And we always know of teachers saying to a boy "Stop crying. Men do not cry!" It is not possible to build knowledge dissociating the rational from the mythical, the sensorial, the intuitive and the emotional.

In the History of Mathematics, we recognize, in every moment, the conjugation of the mythical, the sensorial, the intuitive, the emotional and the rational, although this has been conspicuously ignored in Mathematics Education and, indeed, in more traditional, "internalist", treatments of the History of Mathematics.

Fortunately, there has been a resurgence of interest in the intuitive, sensorial (hands-on projects) and in the affective aspects in Mathematics Education, including the spiritual and the mythical. Every aspect of the individual behavior, every internal structuring, reveals both what comes from inside and what is contextualized. Both an inner voice and the motivation from the context and from the environment, in its broad cultural, social, natural senses, are partners in defining individual behavior. This leads to understanding the human condition. How is it possible to understand the generation and acquisition of knowledge by an individual, if not related to his behavior?

The reaction I usually hear about my position is that since Mathematics is the quintessence of rationalism, these concerns are not to be considered when we examine the nature of Mathematics and of Mathematics Education. Indeed, much of the polemics relate to the prevailing acceptance of the superiority of rationality over other manifestations of human behavior.

I propose a much broader approach to understand all the facets of knowledge: its generation, its organization and its transmission and diffusion. This involves rational, emotional, intuitive, sensorial and mythical and spiritual perceptions of reality.

Generally, the individual origin of mathematical thinking is ignored. How do motivation and needs, the emotional and inner feelings, the imaginary and fantasy, the mythical and the spiritual, play a role in building-up Mathematical knowledge? The research of mathematician Klaus Witz faces the question of spirituality[32].

What had Gustave Flaubert in mind when he wrote *Mathematics: the one who dries up the heart?*[33] How can we identify, in the work of Sophus Lie, the reasons for his remarks in a letter written to Bjornson in 1893:

> Without Fantasy one would never become a mathematician, and what gave me a place among the mathematicians of our day, despite my lack of knowledge and form, was the audacity of my thinking.[34]

Many reactions to these remarks are that they are no-sense, since Mathematics is the quintessence of rationalism. This reaction reveals the prevailing acceptance of the superiority of rationality over other manifestations of human behavior. This was one of the main concerns of the mathematician-writer Robert Musil in his masterpiece *The Man Without Qualities*. Commenting on scientists and engineers, the main character Ulrich says:

> Why they do seldom talk of anything but their profession? Or if they ever do, why do they do it in a special, stiff, out-of-touch, extraneous manner of speaking that does not go any deeper down, inside, than the epiglots? This is far from being true

[32] Witz (2007).
[33] Flaubert (1987, trad. it. 35-47).
[34] Stubhaug (2000: 143).

> of all of them, of course, but it is true of a great many;...They revealed themselves to be men who were firmly attached to their drawing-boards, who loved their profession and were admirably efficient in it; but to the suggestion that they should apply the audacity of their ideas not to their machines but to themselves they would have reacted much as though they had been asked to use a hammer for the unnatural purpose of murder.[35]

Musil's opera anticipates the intellectual framework of Nazi Germany, in which he identifies incapacity to tolerate pluralism. Similarly, the reaction against considering motivation and needs, the emotional and inner feelings, the imaginary and fantasy, the mythical and the spiritual in the studies about knowledge, particularly Mathematics knowledge, is the result of the incapability of accepting the different. Indeed, this is a strategy for the exclusion of the different and for contestation of the power structure, euphemistically referred to as the "normal" social, economical and political order.

It is naïve not to recognize that science and technology have been generated to give to the prevailing social, economical and political order a character of normality. Religion and Science also give the prevailing human individual and social behavior a sense of normality. New insights on the religions and in the sciences lead to a process of dismantling and reassembling systems of knowledge. These insights are, in many cases, referred to as abnormal.

5. In the guise of a conclusion

I have discussed, in this and in many other papers, how mathematical ideas, which are basically the explicit or implicit strategies, intrinsic to the intellectual evolution of the species *homo*, are responsible for the main material and intellectual activities of a society. These activities, both material and intellectual, are organized as the ethnomathematics of the cultural group, relying on the sensorial, intuitive, emotional, mythical and rational dimensions of mathematics ideas. This paper faces the threats to sustained human existence, as discussed in the beginning of this paper. To proceed with the same style of development and progress will lead, inevitably, to destruction. Thus, if Mathematics is the basis of civilization, determinant of the current status of Sciences and of the technological, industrial, military, economic and political complexes, we have to question the nature of prevailing Mathematics. Prevailing Mathematics, which is the Mathematics of modern academies and schools, is a *sui generis* formal and abstract organization of basic mathematical ideas, which are the ways of observing, comparing, classifying, ordering,

[35] Musil (1980: 38).

measuring, quantifying and inferring of the peoples of Antiquity, particularly from Mesopotamia and the Mediterranean Basin, particularly Greeks from 600 BC through 400 AD. If we accept, as I do, that mathematical ideas are the best strategies for the main material and intellectual activities of a society, we are faced with the challenge of proposing a new organization of mathematical ideas. In other words, a new approach to knowledge, particularly to mathematical knowledge, as the support of a new status of human relations and of a renewed economical, industrial and technological sectors. The role of a new mathematics in the search for this new economic order is undeniable. It will even be possible to imagine the emergence of a "soft mathematics", as expressed by Keith Devlin[36]; or an "algebra of knowledge", where the transfer of knowledge from one individual to another does not obey the principle of *al-jabr* (transposition) and *al-muqabala* (reduction). To discuss this need and the possibilities of coping with it is the object of this paper.

To examine other approaches to knowledge, particularly to mathematical knowledge, in different cultures and in different times, we look into mathematical ideas through what might be called "cultural lens". This is helpful in understanding the nature of prevailing Mathematics and may suggest new directions of research. As a result, we may better understand the implications of mathematical research, its contents and pedagogical methodologies.

The answer to the threats to sustained human existence is to develop concepts of knowledge and of human relations which rejects inequity, arrogance and bigotry, essential steps for avoiding violations of Peace in its four basic dimensions: military peace, social peace, environmental peace and inner peace.

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[36] Devlin (1997: 283).

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Les theories cognitives en didactique des mathematiques: lesquelles et pourquoi?

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Les recherches sur l'enseignement des mathématiques se sont développées à partir des années 1960-1970 pour répondre à la situation entièrement nouvelle de l'enseignement des mathématiques, marquée par trois grands changements. Le premier est ce qui alors a été appelé "a massification" de l'enseignement secondaire, imposant à tous les élèves d'une même classe d'âge une formation mathématique plus longue et plus importante. Le second a été un renouvellement complet des contenus enseignés, pour tenir compte à la fois du développement des connaissances mathématiques et de l'apparition de nouvelles demandes sociales en matière de formation. Le troisième a été une demande institutionnelle très forte de formation pour préparer les enseignements à ce renouvellement. Le développement des recherches sur l'enseignement des mathématiques s'est donc fait sur plusieurs fronts, et le recours à des théories cognitives concernant les processus d'acquistion de connaissance, s'est rapidement imposé. C'est ainsi qu'on a eu successivement recours à trois types de théories cognitives, contradictoires entre elles des points essentiels concernant les processus cognitifs de compréhension. La théorie piagétienne centrée sur la construction des concepts, celle de Vygotski centrée sur les différentes pratiques de discours en langue naturelle, et celle de Peirce qui est à la fois pragmatique et sémiotique. Naturellement on propose aussi des versions améliorées de ces théories, ou des montages "conceptuels" qui en seraient la synthèse.

La grande diversité des théories cognitives dont les recherches sur l'enseignement des mathématiques se réclament maintenant, et l'exigence d'en

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"choisir" une pour commencer un travail de recherche conduisent à s'interroger sur ce qu'on appelle une "théorie" en didactique des mathématiques. Pour mener cette réflexion nous partirons de deux contasts que le développement des recherches depuis les années 1970 ont mis massivement en évidence Il y a tout d'abord l'ampleur et la persistance des problèmes de compréhension dans l'apprentissage des mathématiques, pour la très grande majorité des élèves aussi bien au Primaire, au Collège, qu'au Lycée. Il suffit de consulter les enquêtes et les évaluations nationales et internationales, qui n'ont cessé de se multiplier et de prendre de l'importance depuis les années 1990, pour le vérifier. Il y a ensuite le fait que l'enseignement des mathématiques est un monde éclaté. Cet éclatement de l'enseignement se manifeste de deux manières. D'une part, à travers les ruptures bien connues des enseignants, qui apparaissent dans le passage d'un cycle à l'autre, tout au long du curriculum. Ces ruptures tiennent en partie à des changements d'exigence mathématique avec une demande de plus en porte de justification ou de démonstration. D'autre part, dans le fait que les mathématiques enseignées recouvrent des domaines différents : le calcul arithmétique (les entiers, les décimaux, les rationnels, etc.), l'algèbre (calcul littéral, résolution d'équations), la géométrie (géométrie plane, géométrie dans l'espace), l'analyse (fonctions), les statistiques, les probabilités, etc.

C'est par rapport à ces deux constats, qui semblent difficiles à récuser, que nous allons nous essayer de répondre à la question posée dans le titre de cet article. Etant données l'ampleur et la persistance des problèmes de compréhension, les théories cognitives auxquelles on recourt pour justifier des choix didactiques dans l'organisation de séquences d'activités, offrent-elles une description pertinente des processus de compréhension et d'acquisition de connaissances en mathématiques? En réalité, la premiere question qu'il faut se poser pour y répondre est celle-ci. Quels sont les problèmes spécifiques de compréhension que l'apprentissage des mathématiques soulève et que les élèves ne rencontrent pas dans les autres domaines de connaissance scientifique? Etant donné l'éclatement de l'enseignement des mathématiques, les théories élaborées pour un domaine particulier des mathématiques enseignées (les décimaux, l'introduction de l'algèbre, l'analyse, etc.) peuvent-elles être étendues aux autres domaines des mathématiques? Et si, oui, qu'est-ce qui permet leur généralisation?

Dans les deux premières sections de cet article, nous expliquerons brièvement pourquoi et en quoi les connaissances mathématiques requièrent le développement de processus cognitifs spécifiques, qui ne sont pas nécessaires pour comprendre et apprendre dans les autres domaines de connaissance. Puis nous présenterons l'analyse du fonctionnement cognitif de la pensée et de l'activité mathématiques, que nous avons été progressivement conduit à faire, en partant de la situation épistémologique particulière des mathématiques par rapport aux autres domaines de la connaissances scientifique. Dans une quatrième section nous interrogerons sur le statut de cette analyse: théorie, méthode, ou programme de recherche? Dans la dernière section, consacrée aux deux faces de l'activité mathématique, nous apporterons des éléments de réponse à la deuxième question. sur la portée réelle des théories cognitives utilisées en didactique. Locale ou globale? Autrement dit, faut-il autant de théories que de domaines mathématiques et de cycles d'enseignement ou, au contraire, une théorie cognitive doit-elle concerner ce qui fait que l'activité mathématique est fondamentalement la même, quel que soit le domaine enseigné? Et ne serait-ce pas là la première condition pour qu'une théorie cognitive permette de changer la situation générale concernant l'ampleur et la persistance des problèmes de compréhension dans l'apprentissage des mathématiques?

1. Trois distinctions clés pour l'analyse cognitive des connaissances scientifiques

Pour analyser les processus cognitifs qui déterminent le développement et l'acquisition de connaissances scientifiques, il faut recourir à une triple distinction. Celle entre un objet et sa représentation, celle entre une représentation et le système qui permet de la produire, et celle entre un objet et le contenu d'une représentation.

La première distinction répond à l'exigence épistémologique fondamentale, qui a été formulée par Platon et reprise dans toutes les théories de la connaissance. Il ne faut jamais confondre l'objet et sa représentation[1]. Mais cette exigence soulève la question de savoir ce qui permet de les distinguer. D'un point de vue purement cognitif, le critère est simple:

(1) *Il y a beaucoup de représentations possibles différentes pour un même objet, alors qu'il ne peut pas y avoir plusieurs objets différents pour une même représentation.*

La deuxième distinction porte sur la diversité des types de représentations possibles d'un même objet: un énoncé dans une langue naturelle, l'écriture de nombres dans un système de numération, l'écriture algébrique de relations,

[1] Duval (2009: 84).

des figures géométriques, des schémas, des graphes, des reflets dans un miroir, des photographies, des images obtenues avec un télescope ou un microscope, et même ces images mentales que sont les souvenirs et qui sont associées aux mots désignant des choses perçues. Cette distinction donne un critère pour répondre à la question: combien existe-t-il de représentations différentes possibles d'un même objet?

(2) *Il y a autant de représentations différentes possibles d'un objet qu'il y a de systèmes dont la fonction est de produire des représentations.*

Les systèmes producteurs de représentations sont des systèmes sémiotiques comme les langues naturelles, les systèmes de numérations, les axes gradués permettant de construire des graphes, etc. Mais ce sont aussi des systèmes non sémiotiques comme les instruments scientifiques, ou encore comme les organisations neuronales. Cette seconde distinction est la distinction cognitive la plus fondamentale. Mais elle n'est presque jamais prise en compte dans les théories cognitives concernant le développement et l'acquisition de connaissance. Par exemple, Peirce qui a été le premier à insister sur l'importance des différents de représentation, en explique la compréhension et l'utilisation par les différences subjectives et développementales entre les individus, sans du tout prendre en compte les systèmes qui produisent les représentations. Il suffit de regarder sa célèbre définition des signes: "A sign, or representamen, is something *which stands to somebody* for something *in some respect or capacity*"[2]. Cette définition conduit à expliquer la diversité des représentations par les différences subjectives et développementales entre les élèves, sans prendre en compte la nature des systèmes qui produisent les représentations. De même la qualification des représentations comme "mentales", comme "extérieures" et perceptibles par tous, ou encore comme "sémiotiques", repose sur la confusion entre les trois modes phénoménologiques de production, que Vygotski[3] avait distingués pour le langage, et les systèmes producteurs de représentation.

La troisième distinction est la conséquence immédiate des deux précédentes:

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[2] Nous avons mis en italiques les expressions qui distinguent la définition de Peirce de celle, classique, d'Augustin (Duval (2006a)). La sémio-pragmatique de Peirce a été utilisée dans les recherches didactiques, à partir des années 1990, pour expliquer le caractère individuel et transionnel des représentations mobilisées par les élèves dans la résolution de problème. Peirce (1931: 2.228).

[3] Vygotski (1985).

(3) *Deux représentations d'un même objet qui ont été obtenues en utilisant deux systèmes différents ont des contenus différents.*

La diversité des systèmes utilisés pour représenter les nombres entiers en est l'exemple le plus évident. De manière plus générale, on peut avoir une description verbale d'un objet, une photographie de ce même objet, un dessin schématique de cet objet, etc. Le célèbre montage photographique de Kosuth "une est trois chaises" en est une parfaite illustration[4]. Bien qu'elles renvoient au même objet, les trois représentations photographiées ne montrent du tout pas la même chose. Leurs contenus respectifs sont totalement différents. La troisième distinction peut paraître évidente lorsqu'il s'agit de choses que l'on peut percevoir directement. Elle ne l'est plus du tout lorsqu'il s'agit d'objets qu'il n'est pas possible de percevoir, comme nous allons le voir pour les objets mathématiques.

Ces trois distinctions montrent que, pour comprendre les processus de développement et d'acquisition des connaissances scientifiques, toutes les représentations doivent être analysées en fonction du système qui a permis de les produire. Car leur contenu des représenations dépend autant du sytème qui les produit que des objets qu'elles représentent.

2. Les deux points cruciaux pour l'analyse cognitive de la pensée et de l'activité mathématique

D'un point de vue épistémologique, les mathématiques constituent un domaine à part. Car les moyens d'accès aux objets étudiés sont différents des moyens d'accès communs aux autres sciences. C'est la raison pour laquelle la nature même de l'activité mathématique repose sur des processus cognitifs qui sont purement sémiotiques et non pas empriques et "conceptuels".

2.1. La production de représentations sémiotiques est l'unique moyen d'accès aux objets mathématiques

On n'accède pas aux objets mathématiques directement par la perception ou en utilisant des instruments produisant des représentations non sémiotiques qui étendent le champ des phénomènes observables au delà la perception immédiate à l'œil nu. On accède aux objets mathématiques en produisant des représentations sémiotiques. Prenons l'exemple des nombres naturels, dont

[4] Duval (2006b: 591).

les plus petits peuvent représentés par des collections d'objets. L'accès aux nombres naturels commence avec une opération de comptage, opération qui consiste à désigner un à un chacun des objets d'une collection, en utilisant une liste de mots dans un ordre qui ne doit pas varier. *Cette production verbale orale est nécessaire pour un premier accès aux nombres.* Et si des opérations additives peuvent être effectuées en séparant ou en regroupant spatialement les objets comptés, c'est à dire en effectuant des comptages avant puis après, la puissance de calcul a exigé le développement de systèmes d'écriture qui utilisent seulement quelques caractères, dont ce signe à part "0", *lequel n'intervient dans aucun comptage*[5]. Ces systèmes ne sont pas nécessaires pour le comptage et pour les opérations additives qui peuvent se faire uniquement en production orale. En revanche, ils le deviennent pour effectuer les opérations multiplicatives.

De manière plus générale le développement des mathématiques est lié à l'invention de nouveaux systèmes sémiotiques ou à leur perfectionnement. L'exemple le plus spectaculaire est la révolution sémiotique qui s'est produite, en mathématiques, aux XVIe et XVIIe siècle, en moins de 150 ans. Il y a eu l'invention d'une écriture littérale pour exprimer des relations d'égalité ou d'inégalité entre des grandeurs et, dans la foulée, l'invention d'un système de représentations graphiques en utilisant une règle de correspondance entre un point et un couple de nombres sur deux axes gradués. L'invention de ces deux systèmes ont marqué l'émergence de l'algèbre et celle de l'Analyse[6]. Notons ici que l'invention du système cartésien de représentation graphique constitue un type visualisation mathématique qui est totalement différent de la visualisation proprement géométrique qui s'était développée avec Euclide. La prise en compte de leurs différences, à la fois sémiotique et cognitive, est d'autant plus importante que ces *deux types de visualisation spécifiquement mathématiques,* ont été très vite utilisés dans l'enseignement pour introduire des notions mathématiques.

2.2. L'activité mathématique consiste dans la transformations de représentations sémiotiques en d'autres représentations sémiotiques

En mathématiques le rôle fondamental des représentations sémiotiques n'est pas de donner accés aux objets mathématiques, c'est à dire aux nombres, aux fonctions, aux propriétés géométriques, etc., il est surtout de pouvoir être transformées en d'autres représentations sémiotiques. Sans cela, aucune

[5] Duval (2006b: 594).
[6] Duval (2011a: 24-25).

forme de calcul, arithmétique, algébrique, intégral, ou formel, ne serait possible.

Frege est le premier à s'être intéressé au mécanisme sémantico-logique de la transformation de représentations sémiotiques en d'autres représentations sémiotiques. Cherchant à comprendre la différence entre la stérilité des tautologies "a = a" et la fécondité d'égalités de type "a = b", qui permettent au calcul de produire des résultats nouveaux, il a décrit le calcul comme *une opération de substitution, salva veritate, d'un signe à un autre*, d'une expression à une autre[7]. Et, pour en expliquer le mécanisme sémantico-logique, il a introduit sa célèbre distinction entre le sens d'un signe (*Sinn*) et la référence de ce même signe (*Bedeutung).* Ainsi il n'y a pas de substitution possible à partir de tautologie "a = a" qui portent sur deux occurrences du même signe. Le contenu du signe et l'objet qu'il représente sont les mêmes dans les deux occurrences. Au contraire, dans l'égalité "a = b" les contenus des deux signes "a" et "b" sont différents, et si l'égalité est vraie, *c'est à dire si ces deux signes réfèrent au même objet, alors on pourra substituer a à b.* C'est le mécanisme fondamental du calcul en mathématiques qui permet d'effectuer des substitutions logiquement valides et mathématiquement productives de résultats nouveaux. On peut reformuler la célèbre distinction de Frege dans la proposition suivante.

(4) *Si entre deux expressions "a" et "b", l'expression complète "a = b" est vraie, c'est à dire si ces expressions de sens différents représentent le même objet, alors* ON PEUT SUBSTITUER *a à b* DANS UNE AUTRE *EXPRESSION COMPLETE COMPORTANT LE TERME.*

Tous les débats auxquels cette distinction a donné lieu s'en sont tenus à la question de la vérité de l'antécédent de cette implication[8], en oubliant que la distinction entre sens et référence visait à expliquer la substitution de a à b dans une deuxième expression complète comprenant comme l'un de ses termes soit "a" soit "b".

Cet énoncé peut être généralisé, comme on va le voir, à toute l'activité mathématique, c'est à dire à tout ce qui en mathématique ne relève pas directement d'un calcul mais qui est cependant nécessaire pour comprendre, justifier ou utiliser des calculs dans la résolution de problème. Car toutes les

[7] Frege (1971).

[8] On s'est interrogé, à la suite de Russell, sur le caractère absurde, faux ou acceptable d'une expression complète dans le cas où l'un de ses termes est une représentation vide ou fausse. Mais le mécanisme sémantico-logique de Frege porte sur la possibilité d'une substitution dans une deuxième expression complète pour assurer la progression du calcul.

formes d’activité mathématique relèvent de ce processus de substitution, l’une à l’autre, de deux représentations sémiotiques différentes d’un même objet. Ce processus est indépendant de l’acquisition de concepts.

3. L’analyse du fonctionnement cognitif de l’activité mathématique et les problèmes de compréhension dans l’apprentissage des mathématiques

L’activité mathématique consistant dans la transformation par substitution de représentations sémiotiques en d’autres représentations sémiotiques (ci-dessus (4)), et toutes les représentations sémiotiques dépendant du système qui les produit (ci-dessus (2)), le premier outil théorique d’analyse doit être une classification cognitive de tous les systèmes sémiotiques utilisés en mathématiques. Cette classification va permettre de distinguer les deux grands types de transformations des représentations sémiotiques qu’on retrouve dans la résolution des problèmes mathématiques comme dans l’application de connaissances mathématiques aux problèmes rencontrés dans la réalité. Les difficultés systématiques et récurrentes de compréhension dans l’apprentissage des mathématiques portent sur ces deux grands types de transformation.

3.1. La classification cognitive des systèmes sémiotiques utilisés en mathématiques

Tous les systèmes sémiotiques utilisés dans le cadre des mathématiques doivent permettre de transformer des représentations sémiotiques en d’autres représentations (ci-dessus (4)). Car leur utilisation doit permettre de parvenir à des résultats neufs ou à la solution de problèmes, indépendamment de tout recours à des sources externes d’informations ou de données. Il existe plusieurs types de systèmes sémiotiques qui satisfont à cette condition fondamentale, mais qui remplissent des fonctions cognitives très différentes. Nous avons ainsi distingué, d’une part, les systèmes discursifs permettant le raisonnement ou le calcul et les systèmes producteurs de représentations synoptiques permettant de saisir d’un seul regard toutes les relations internes d’un objet, ou les variations d’un phénomène, d’une mise en correspondance de deux ensembles de nombres, etc. Et, d’autre part, nous avons distingué, les systèmes multifonctionnels qui remplissent d’autres fonctions que celle de traitement mathématique, telles que la communication ou l’objectivation, et les systèmes monofonctionnels, dont la création a marqué des seuils dans le développement historique des mathématiques, et qui permettent d'algorithmiser les transformations de représentations sémiotique. Nous avons appelé "registres" tous les systèmes sémiotiques qui satisfont à la condition fondamentale de substituabilité définie par Frege et qui remplissent l'une ou l'autre de ces différentes fonctions cognitives[9].

Pour établir la classification nous avons croisé deux partitions différentes des registres. Le première partage les registres en registres discursifs (la langue naturelle, les écritures algébriques, etc.) et registres non discursifs (toutes les représentations dont l'organisation ne relève pas d'une appréhension successive et de règles syntaxiques). Le deuxième les partage les registres dans lesquels les traitements ne sont pas algorithmisables et registres dans lesquels les traitements sont au contraire des algorithmes. On voit ainsi apparaître un clivage entre la langue naturelle et le langage algébrique, entre la visualisation géométrique et la visualisation des graphiques cartésiens[10]. Les registres de représentation utilisés en mathématiques se répartissent donc en quatre grandes classes.

Dans cette classification nous n'avons pas pris en compte toutes les représentations sémiotiques ou non sémiotiques qui sont utilisées dans l'enseignement et que l'on retrouve dans les manuels, parce qu'elle ne satisfont pas à la condition fondamentale de substituabilité énoncée par Frege.

3.2. Les deux processus fondamentaux de l'activité mathématique

Ils correspondent aux transformations de représentations faites en restant dans le même registre et à celles faites en passant d'un registre à un autre. Nous les avons appelés respectivement les "traitements" (mathématiques) et les "conversions".

Ce sont les traitements qui constituent les seules transformations pertinentes du point de vue mathématique, puisqu'ils permettent d'expliquer un résultat ou de le démontrer. Ils dépendent des possibilités de transformation de représentations sémiotiques qui sont propres à chacun des registres. Autrement dit, chaque registre permet d'effectuer des traitements qui lui sont spécifiques, et qui ne peuvent pas être effectués dans un autre registre. Le registre privilégié est, évidemment, celui des écritures symboliques permettant les différentes formes de calcul, numérique, algébrique, etc. C'est en se référant à ce registre que Frege a analysé l'activité mathématique comme ac-

[9] Duval (1995a).

[10] Duval (2006c: 110; 201: 118).

tivité consistant à substituer des expressions les unes aux autres, *salva veritate*. Mais en géométrie élémentaire, où les raisonnements se font dans la langue naturelle, les raisonnements qui consistent à substituer des propositions les unes aux autres sont plus proches du calcul que de l'argumentation[11]. La substitution, qui se fait en fonction d'un énoncé tiers, c'est à dire d'un théorème, d'une définition ou d'un axiome, est toujours localement valide et mathématiquement productive, comme pour les calculs. Et nous avons pu voir que la prise de conscience de ce mécanisme de substitution dans la langue naturelle était vécue par les élèves comme une découverte de la force des preuves mathématiques contre toute évidence perceptive[12]. Le mécanisme de substitution pour les figures géométriques est radicalement différent. Il porte sur opérations visuelles de reconnaissance d'autres unités figurales que celles qui s'imposent perceptivement au premier regard. Ainsi, les opérations de reconfiguration permettent de montrer l'égalité de deux surfaces par des décompositions et des recompositions d'unités figurales 2D des deux surfaces comparées. L'exemple le plus célèbre est celui des preuves chinoise, hindhoue et autres, du théorème de Pythagore[13].

Convertir une représentation d'un objet qui a été produite dans un registre consiste à lui substituer une autre représentation du même objet que l'on produit dans un autre registre. C'est ce type de transformation qui est le plus complexe et le plus paradoxal d'un point de vue cognitif, et c'est aussi celui qui est le plus fondamental pour comprendre les concepts en mathématiques et, surtout, pour résoudre des problèmes. Les conversions de représentations sémiotiques sont des changements de registre orientés vers le choix du registre dans lequel on va pouvoir effectuer un traitement. Les conversions présentent deux caractéristiques cognitivement importantes:

- La représentation produite par conversion a un contenu totalement différent de la représentation de départ, bien que les deux représentent le même objet mathématique (ci-dessus (3)). C'est cette différence de contenu qui détermine la distance cognitive entre deux représentations d'un même objet. Et cette distance cognitive varie en fonction du registre de départ et le registre d'arrivée.
- Les conversions sont des opérations de substitution qui sont à la fois indépendantes de tout traitement intermédiaire et du recours à un concept. Cette indépendance se traduit dans le fait que, pour un

[11] Duval (2000: 157).
[12] Duval (1991; 2007).
[13] Duval (1995b).

couple de registres donné, les élèves pourront réussir les conversions dans un sens et échouer presque systématiquement pour la conversion inverse[14].

Autrement dit, la conversion des représentations relève d'un processus cognitif de reconnaissance spontanée du même objet dans deux représentations dont les contenus sont différents parce leur production relève de deux systèmes différents.

La prise en compte des conversions et des traitements conduit à une définition cognitive de l'activité mathématique différente de celle Frege (ci-dessus (4)). Car celle de Frege ne vaut que pour les traitements, c'est à dire pour les seules transformations importantes, si on se limite aux résultats mathématiques et à la manière de les démontrer.

(5) *Toute activité mathématique mobilise, soit explicitement soit implicitement, aux moins deux registres de représentations sémiotiques*

En géométrie élémentaire, par exemple, ou dans un travail sur les fonctions, on mobilise, simultanément ou alternativement, deux registres, l'un jouant un rôle heuristique ou un rôle de contrôle par rapport à l'autre qui est le registre dans lequel on effectue les traitements. En dehors des tâches de rédaction, la langue naturelle est le deuxième registre qui est sollicité pour expliquer oralement ce qu'on écrit au tableau.

3.3. Les conversions sont la première source d'incompréhension des mathématiques

Les difficultés systématiques et récurrentes de compréhension dans l'apprentissage des mathématiques ne viennent pas des concepts mathématiques, mais de la situation épistémologique à part, des mathématiques.

En mathématiques, à la différence des autres domaines de connaissance scientifique, l'utilisation de représentations sémiotiques est l'unique moyen d'accès aux objets étudiés (ci-dessus 2.1). Et cela soulève ce que nous avons appelé le paradoxe cognitif des mathématiques[15]. *Comment ne pas confondre un objet et sa représentation, c'est à dire un objet et le contenu de la représentation qu'on en donne (ci-dessus, (3))* quand il n'y a pas d'autre accès à l'objet lui-même que l'utilisation d'une autre représentation sémiotique? En

[14] Duval (1988).
[15] Duval (2006c).

physique, en chimie, en géologie, en botanique, il n'y a pas de paradoxe cognitif, parce qu'il y a toujours un accès perceptif direct aux objets étudiés en allant sur le terrain, ou un accès par l'utilisation d'instruments "scientifiques" en laboratoire, qui produisent *des représentations non sémiotiques*[16]. C'est pourquoi les contenus de deux représentations d'un même objet mathématique, produites dans registres différents sont inévitablement, pour la très grande majorité des élèves deux objets mathématiques différents.

D'un point de cognitif, le critère de compréhension en mathématiques est la reconnaissance immédiate d'un même objet dans des représentations dont les contenus n'ont rien de commun. Car toute résolution de problème, et plus encore celle des problèmes dit "concrets" ou "réels", exige qu'on puisse reconnaître pour chaque donnée du problème deux représentations différentes: une expression verbale et l'expression numérique ou l'expression algébrique correspondante[17], ou encore une description verbale et le symbole de relation correspondant. Pour les problèmes de géométrie, cela est encore plus délicat: il faut reconnaître à partir d'une description verbale, les sous-figures correspondantes dans la figure donnée pour le problème, et les sous-figures correspondant à la propriété à utiliser. LA TACHE COGNITIVE demandée aux élèves dans une résolution de problème est d'abord la suivante:

> Une seule des deux représentations étant présentée dans l'énoncé pour chaque donnée du problème, *IL FAUT PRODUIRE les representations correspondantes* permettant d'écrire l'opération arithmétique à effectuer, ou de choisir la propriété géométrique à appliquer.

Cela présuppose, bien évidemment, que si les deux représentations étaient présentées en même temps, ils soient capables de reconnaître dans des contenus différents la même information. Ce qui est la situation après coup, lorsque l'enseignant explique la solution du problème. Autrement dit, *comprendre, c'est d'abord reconnaître*. Les phénomènes d'incompréhension observés dans les situations de résolution de problème, restent les mêmes aux différents niveaux de l'enseignement, quels que soient les "concepts" à appliquer. Ils sont de deux types. Soit *un blocage et un abandon rapide de toute activité de recherche*, ou des erreurs révélant des onfusions ininterprétables. Soit une fausse reconnaisance des unités discursives ou figurales et des unités

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[16] La différence épistémologique et cognitif qui sépare les représentations sémiotiques et les représentations non sémiotiques, voir l'annexe.

[17] Les expression numériques et les expressions algébriques sont des expressions composées de deux types différents de signes: des chiffres ou une lettre qui réfèrent à un nombre ou à un ensemble de nombres, et un symbole d'opération. Les symboles de relation permettent de former des expressions complètes (égalités numériques, équations, inéquations).

symboliques à mettre en correpondance. *Cette fausse reconnaissance est d'autant plus prégnante qu'elle serait en réalité une bonne reconnaissance si l'on était pas dans le domaine des mathématiques!* Et cela conduit aux erreurs systématiques et récurrentes que l'on retrouve à tous les niveaux de l'enseignement. Les fausses reconnaissances, tant qu'elles persistent, rendent incompréhensible toute explication mathématique, et elles rendent de plus en plus difficile tout réel progrès dans l'apprentissage des mathématiques. En classe, on peut contourner localement ces phénomènes d'incompréhension en faisant travailler les élèves de manière interactive pour résoudre un problème. Mais on s'aperçoit très vite qu'il n'y a aucun tranfert sur d'autres situations relevant du même type de problème. Autrement dit, les phénomènes d'incompréhension persistent même après ce travail en commun et son "institutionnalisation".

La résolution de problème en mathématiques perme donc de voir que le premier seuil de seuil compréhension est la conversion. Et ici le critère de reconnaissance doit être pris au sens fort. La reconnaissance doit se faire, par exemple, pour tous les problèmes additifs, et non pas seulement pour certains, ou pour toutes les situations possibles d'application d'un théorème. *Cela signifie qu'il n'y a pas de compréhension, aux yeux même des élèves, s'ils n'acquièrent pas une capacité personnelle d'initiative et de contrôle face à de nouvelles situations de problèmes*. Ils se retrouvent chaque fois dans le même embarras. Le second seuil est celui de l'exigence mathématique: comprendre, c'est pouvoir justifier un résultat ou le choix d'une procédure. Or c'est le premier seuil cognitif que la plupart des élèves ne parviennent pas à franchir. Il tient à la manière mathématique de travailler, et non pas d'abord aux concepts mathématiques.

3.4. Développer la coordination entre registres pour faire franchir le premier seuil de compréhension en mathématiques

Le processus de reconnaissance d'un même objet mathématique dans des représentations produites dans deux systèmes sémiotiques repose la mise en correspondance des unités de sens qui constituent les contenus respectifs de ces représentations[18]. Le développement de la coordination cognitive entre deux registres exige donc un travail specifique pour faire prendre conscience des unités de sens pertinentes dans les énoncés *en regard avec* celles pertinentes avec les écritures algébriques, ou encore en regard avec les "visual

[18] Duval (2006c: 125).

features" des graphes, ou encore avec les unité figurales d'une figure géométrique, etc. Car c'est seulement par la mise en correspondance des unités de sens constituant les contenus respectifs de deux représentations que l'on peut reconnaître si ces représentations représentent sont, ou ne sont pas, deux représentations d'un même objet[19]. Un élève franchit le premier seuil de compréhension, lorsqu'il peut reconnaître spontanément un même objet mathématique, dans les deux sens de la conversion et quelles que soient les variations effectuées dans le contenu de l'une des deux représentations.

La situation cognitivement la plus complexe est celle où l'un des deux registres mobilisés est la langue naturelle. Là, il faut distinguer plusieurs cas, selon que la langue naturelle est le registre de départ ou le registre d'arrivée, et selon que l'autre registre est celui des écritures algébriques ou un registre de visualisation. Par exemple, dans le cas où la langue est le registre de départ et où le registre d'arrivée est celui des écritures symboliques, il est nécessaire de recourir à des représentations sémiotiques qui ne dépendent d'aucun registre. La fonction de ces représentations auxiliaires est alors de réduire la distance cogntive entre le contenu d'un énoncé verbal et le contenu de l'égalité numérique ou de l'équation dont la résolution donnera la solution du problème. Le problème que l'utilisation didactique de repréntations auxiliaires soulève est de savoir si elles permettent de remplir cette fonction[20]. Un cas complétement différent est celui où la langue naturelle est le registre d'arrivée, le registre de départ étant une figure géométrique. Dans ce cas, la prise de consciences des différentes opérations discursives permettant de designer des unités figurales devient alors nécessaire, comme par exemple pour écrire le programme de construction d'une figure[21].

4. Les registres de représentation sémiotique: une théorie, une méthode, un outil d'analyse ou un programme de recherche?

Tout d'abord la notion de registre a été développée comme UNE METHODE D'OBSERVATION ET D'EXPLORATION des phénomènes de compréhension et d'incompréhension en mathématiques. La notion de registre permettait de ne plus considérer les représentations sémiotiques isolément, c'est à dire à partir des objets représentés, mais *comme l'ensemble de toutes les représentations possibles de même type que l'on peut produire et qui peuvent avoir un sens mathématique*. On peut ainsi considérer toutes les représentations graphiques

[19] Duval (2011a: 48-51).
[20] Duval (2005); Didierjean (1997).
[21] Duval (2015).

que l'on peut produire sur un plan cartésien et distinguer leur variation perceptivement significatives: une ou plusieurs droites, des courbes, des formes géométriques (cercles, paraboles), etc. Dans l'étude de 1988, on s'était limité aux droites. Mais le même travail a été ensuite été réalisé pour les fonctions quadratiques, etc. Ensuite la définition cognitive de l'activité cognitive impliquait que l'on prenne aussi en compte un autre registre (ci-dessus, (5)). Car autrement, il est impossible de discerner les différents "visual features" mathématiquement pertinents qui fusionnent nécessairement dans le contenu global de la perception. Pour les graphiques cartésiens, l'écriture algébrique des relations s'imposait. Mais le travail décisif a été celui sur la découverte du raisonnement déductif (1991). Le problème de compréhension auquel les élèves se heurtaient dans la langue naturelle était double: l'incapacité à distinguer un énoncé et sa réciproque et l'incapacité à voir que le sens d'un proposition tient autant à son statut qu'à son contenu. Il a fallu trouver un type de représentation que les élèves puissent construire eux-mêmes et qui ne soit ni la langue naturelle ni les figures géométriques. Car l'articulation entre le langage et les figures géométriques est une source profonde de difficultés pour les élèves. Nous avons alors choisi les graphes propositionnels qui étaient utilisés, dans la modélisation de la compréhension des textes, pour représenter les différents niveaux de développement d'un récit. Ce sont ces études qui nous ont permis de tester la pertinence et la fécondité de la notion de registre comme méthode d'exploration et d'observation.

Ensuite nous avons aussi utilisé les registres comme UN OUTIL D'ANALYSE des problèmes et des activités mathématiques donnés aux élèves, ainsi que des productions des élèves. Car, bien évidemment, l'analyse et l'interprétation des productions des élèves dépendent de la manière dépendent des tâches que l'on leur demande de faire. On a pu ainsi développer une méthode d'analyse cognitive, et non pas mathématique, des problèmes et de l'organisation des séquences d'activité donnés aux élèves. Cette analyse repose sur le principe suivant:

(6) *Il faut commencer par SEPARER LES DIFFERENTES CONVERSIONS ET LES TRAITEMENTS à effectuer pour arriver à la solution du problème, ou requis par les différentes activités d'une séquence didactique.*

Autrement dit, il faut commencer par décomposer les activités mathématiques en une suite de tâches cognitive de conversions et de traitements. Naturellement, pour les conversions, il est nécessaire de préciser les deux registres mobilisés et le sens de conversion. Et il est également nécessaire de préciser le registre dans lesquels les traitements sont effectués. Cela permet non seulement de décomposer les démarches de réponse des élèves, et donc d'identifier les points précis où des blocages et des fausses reconnaissances se sont produits, mais aussi les difficultés de traitements rencontrés dans les raisonnements en langue naturelle ou dans l'exploration heuristique des figures. En appliquant cette procédure d'analyse aux problèmes proposés dans le cadre de tous les travaux de recherche, on est souvent conduit à des interprétations des difficultés et des réussites, qui sont souvent différentes de celles avancées par les auteurs de ces travaux.

La classification des registres de représentation a ouvert UN PROGRAMME DE RECHERCHE qui est encore loin d'être achevé. Nous ne mentionnerons que trois chantiers. Le premier est celui la visualisation en géométrie. Elle se fait contre toute reconnaissance perceptive des formes 2D ou 3D, car elle exige leur déconstruction dimensionnelle en unités figurales 1D ou 0D[22]. Le deuxième chantier est celui de l'emploi de la langue naturelle qui est triple: emploi mathématique pour une explication ou un raisonnement (termes et propositions) souvent lié à une production écrite, emploi pour une fonction d'objectivation pour soi dans la conduite d'une action et dans la fixation de ce que l'on remarque, et emploi pour une fonction de communication orale entre l'enseignant et les élèves[23]. En classe, la communication orale qui est le mode principale d'interactions entre les élèves et avec l'enseignant, tend à mélanger de manière indistincte ces trois types d'emplois radicalement différents. Le troisième chantier est celui de la résolution de problèmes. Non seulement elle reste une boîte noire pour la très grande majorité des élèves tout au long du curriculum, mais elle est le point de cristallisation majeur de l'incompréhension en mathématiques. L'analyse des activités en termes de registres permet de renverser la perspective dans laquelle les recherches sur cette question ont été menées jusqu'à maintenant[24].

Enfin les registres de représentations sont UNE THEORIE DU FONCTIONNEMENT COGNITIF DE LA COMPREHENSION EN MATHEMATIQUES. Mais la théorie ne porte pas sur l'importance donnée aux représentations sémiotiques, ni même sur la notion et la distinction des registres. Elle porte sur la description cognitive de l'activité mathématique par deux types de transformations de représentations sémiotiques en d'autres représentations sémiotiques. Trois points apparaissent alors théoriquement essentiels.

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[22] Duval (2005); Duval (2013a); Duval (2015).
[23] Duval (2000:149-150).
[24] Duval (2005: 2013).

- Il n'y a pas un type de représentation qui serait plus important qu'un autre pour pouvoir comprendre en mathématique. La compréhension commence quand on est capable de voir les correspondances entre les contenus des représentations de deux registres différents (ci-dessus, 3.4)). Et cela vaut aussi pour les représentations auxiliaires iconiques. Les représentations auxiliaires iconiques n'aident que si les élèves peuvent établir des correspondances entre leur contenu et les contenus des représentations entre lesquelles elle doivent réduire la distance cognitive.
- La coordination des registres est la condition requise pour comprendre en mathématiques, c'est à dire pour devenir capable d'initiative et de contrôle dans la résolution de problèmes. La coordination de différents systèmes de fonctionnement est d'ailleurs l'axe principal du développement cognitif de l'enfant. Ainsi les deux premiers seuils importants sont la coordination de la vision et des gestes, et celui plus complexe de la reproduction vocale des phonèmes de la langue entendue et de la compréhension des mots employés dans l'environnement parlant dès 8-10 mois. La coordination de la reproduction vocale des phonèmes et celle mots reconnus expression d'un sens marque les débuts de la parole entre 16 et 24 mois. L'apprentissage des mathématiques requiert des coordinations analogues entre les registres.
- La caractérisation cognitive des différents registres permet d'identifier les facteurs qui déterminent la prise de conscience du processus de conversion et celle des moyens spécifiques de transformations des représentations sémiotiques à l'intérieur de chaque registre.

En tant que théorie, les registres de représentation visent à analyser tous les phénomènes observables relatifs à ces points fondamentaux et à définir les variables didactiques pertinentes dans le choix des activités, pour favoriser le développement, chez les élèves, des “gestes intellectuels” qui sont propres à l'activité mathématique et qu'on ne retrouve pas dans les domaines de connaissance.

5. Les deux faces de l'activité mathématiques, et les impasses de l'enseignement

Enseigner les mathématiques à tous les élèves jusqu'à 16 ou 17 ans, c'est à dire avant toute spécialisation de l'enseignement en filières à préorientation professionnelle, implique que l'on prenne en compte d'autres points de vue que le seul point de vue mathématique. Lesquels? Dans quelle mesure ces autres points de vue sont-ils à la fois nécessaires pour l'acquisition de connaissances en mathématiques et compatibles avec les exigences propres à toute connaissance mathématique? Ces questions sont les questions préalables à toute recherche sur l'enseignement des mathématiques. Elles s'imposent également pour comparer les « théories » qui se rapportent à ce domaine très diversifié et complexe qu'on appelle en anglais "Mathematics Education". Car ce ne sont pas du tout les mêmes phénomènes qui apparaissent et que l'on étudie, selon le point de vue auquel on se place. Nous en distinguerons deux plus trois. Les deux premiers se rapportent à l'activité mathématique elle-même en tant qu'elle constitue un type de connaissance qui, épistémologiquement, est différent de tous les autres types de connaissance. Les trois autres points de vue se rapportent aux principales catégories d'acteurs qui interviennent dans l'organisation institutionnelle de l'enseignement des mathématiques et qui constituent la réalité d'un système éducatif. C'est par rapport à ces différents points de vue qu'il faut situer les "théories" pour savoir le champ des phénomènes observables pris en compte et la nature processus décrits. Toute théorie se rapporte nécessairement à un seul point de vue.

5.1. Les deux faces de l'activité mathématique et les points de vue mathématique et cognitif

L'activité mathématique comporte deux faces. Il y a celle, proprement mathématique, centrée sur les concepts, c'est à dire sur tous les résultats acquis en tant que "théorèmes" au cours de l'histoire des mathématiques. Les concepts mathématiques condensent des connaissances sous forme de propriétés, de formules, d'algorithmes de calcul. Leur application à tous les autres domaines de la connaissance conduit à dire que "*le monde est mathématique*" selon le titre d'une collection récemment publiée domaine très diversifié et complexe de ce qu'on appelle en anglais "Mathematics Education". Nous appellerons cette face "*la face exposée*" des mathématiques. L'autre face porte sur la manière de travailler qui est propre aux mathématiques. Car elle exige des manières de voir, de raisonner et d'explorer par le seul jeu de transformations de représentations sémiotiques en d'autres représentations sémiotiques. Et cela va contre les manières de voir, de raisonner qui spontanément mises en œuvre dans les autres domaines de connaissance où le travail se fait en utilisant des instruments scientifiques pour recueillir des données ou faire des expérimentations. Cette face est l'envers de la face exposée des mathématiques. Nous l'appellerons "*la face cachée*" des mathématiques.

Du point de vue mathématique, cette face cachée ne compte pas pour deux raisons simples. Tout d'abord la manière de travailler en mathématiques consiste à utiliser des concepts et on peut la décrire en procédures et en méthodes. L'heuristique, par exemple, a longtemps cherché à dégager des méthodes pour résoudre des problèmes. Et il y a des méthodes de résolution pour les équations. La deuxième raison est *la transparence des représentations sémiotiques,* du moins pour les mathématiciens et ceux qui "comprennent". *Ils reconnaissent d'emblée les objets mathématiques représentés*, sans les confondre avec les représentations sémiotiques produites. Celles-ci sont remarquées comme des "objets phénoménologiques", lorsque, dans leur production écrite ou graphique, il s'agit de regarder leurs "forme" c'est à dire leurs organisations internes, comme c'est le cas en algèbre pour simplifier des expressions.

Du point de vue cognitif, cette face cachée correspond à la manière mathématique de travailler pour résoudre des problèmes, mathématiques ou des problèmes concrets, quels que soient les concepts à utiliser. Car, dans toute résolution de problème, il faut pourvoir reconnaître les concepts pertinents à utiliser. Or comme nous l'avons vu plus haut (3. 3), elle concerne les conditions requise pour franchir le premier seuil de compréhension en mathématiques. *Autrement dit, c'est par cette face cachée que l'on peut réellement accéder à la face exposée des mathématiques*!

L'irréductibilité de la face cachée de l'activité mathématique à sa face exposée, et son importance pour l'apprentissage des mathématiques viennent du statut épistémologique des mathématiques. Cela apparaît immédiatement non seulement dans la comparaison des moyens d'accès aux objets de connaissance en mathématiques et dans les autres sciences, mais aussi dans celle des preuves mathématiques et des preuves dans les autres domaines scientifiques[25]. Or dans les recherches sur l'enseignement des mathématiques, le point de vue d'une épistémologie comparative est négligé au profit d'une épistémologie intra-disciplinaire centrée sur l'histoire des concepts[26]. Ainsi

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[25] Duval (2011b).

[26] La première épistémologie comparative est celle de Kant. La *Critique de la raison pure* (1788) est une théorie des conditions différentes de développement pour la géométrie euclidienne (concept d'espace), pour l'arithmétique (schème du nombre) et pour la physique newtonienne (concepts de temps, de mouvement, de causalité). Le mot « épistémologie » est apparu plus tard, au milieu du XIXème, mais il n'a pris son acception actuelle qu'au début du XXème. L'épistémologie génétique de Piaget a repris les hypothèses et les analyses néo-

l'épistémologie des mathématiques, qui est essentiellement centrée sur l'émergence et le développement des concepts mathématiques laisse évidemment hors champ, le statut épistémologique de la connaissance mathématique par rapport aux types de connaissance scientifique.

5.2. L'enseignement des mathématiques et les points de vue institutionnel, pédagogique et développemental

Ces trois point de vue correspondent respectivement aux trois catégories d'acteurs d'un système éducatif: les responsables institutionnels et les experts, les enseignants dans leurs classes, et… les élèves.

L'enseignement des mathématiques est organisé en fonction *d'un objectif global* de formation, générale ou spécialisée, à atteindre sur un cycle, c'est à dire sur plusieurs années. Deux choix commandent cette organisation. Le premier est celui des connaissances complètes, c'est à dire des connaissances constituant un outil mathématique, qui répondent aux objectifs globaux de la formation. Le deuxième porte sur la décomposition de ces connaissances en concepts et procédures qui vont constituer *des objectifs locaux* d'acquisition, pour chaque année en classe. La méthode suivie pour décomposition est celle d'une analyse régressive. On part de la connaissance complète dont on vise l'appropriation par les élèves à la fin d'un cycle d'enseignement. *On isole LES CONNAISSANCES MATHEMATIQUES PREREQUISES pour la compréhension mathématique de cette connaissance***.** Puis on réitère cette décomposition sur chacune des connaissances apparues comme des prérequis composant la connaissance complète. Cette réitération est faite jusqu'à ce qu'on obtienne des contenus *minimaux ayant encore en sens mathématique*. On obtient ainsi tous les éléments de base à faire acquérir. Ce type de décomposition d'une connaissance en connaissances prérequises est analogue à la démarche *top-down* utilisée en linguistique pour décomposer le sens des phrases en ses éléments constituants. L'organisation de l'enseigneent de l'algèbre au Collège en est une parfaite illustration[27].

L'organisation de l'enseignement sur un cycle se fait alors selon *l'ordre inverse, bottom up, de cette décomposition régressive*. La progression dans

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kantiennes que Brunschvicg avait développées dans *Les Étapes de la philosophie mathématique (1912)* et dans *L'Expérience humaine et la causalité physique* (1922). Dans un long compte rendu de ce dernier ouvrage, Piaget en avait fait explicitement tout son programme de recherches (Piaget 1924). Les concepts dont Piaget a étudié la genèse ne sont ni des concepts mathématiques, ni des connaissances prérequises pour la compréhension en mathématiques.

[27] Duval (2015b).

l’acquisition d’une connaissance mathématique complète est donc mathématiquement et institutionnellement définie comme une succession de connaissances prérequises à acquérir en vue de (*RE)CONSTRUIRE* la connaissance complète correspondant à l’objectif global d’un cycle. Mais cela revient à méconnaître ce fait majeur:

(7) *Ce qui le plus simple ou le plus élémentaire d’un point de vue mathématique est presque toujours cognitivement complexe. Ce qui est mathématiquement simple ou élémentaire n’apparaît comme tel qu’au terme d’un apprentissage, lorsqu’on a pris conscience des manières de voir, de définir et raisonner en mathématiques.*

L’enseignement se fait en classe, du moins jusqu’à présent, sous la conduite d’un enseignant. Il porte sur les objectifs locaux fixés dans le cadre de la progression institutionnelle de l’acquisition d’une connaissance complète. L’enseignant doit donc organiser des situations d’apprentissage, souvent appelées “séquences d’activités” , pour chacun des concepts et des procédures correspondant aux objectifs locaux de l’année. Mais la gestion des situations d’apprentissage repose sur la *communication* entre l’enseignant et les élèves, et entre les élèves. Cette communication est une dimension évidemment orthogonale aux tâches proposées pour l’apprentissage d’une procédure ou la prise de conscience d’une propriété mathématique. Elle se fait selon les différents modes de production propres à la communication et qui traduisent les attitudes et réactions personnelles: l’expression orale avec ses intonations, accompagnée ou non de gestes, les silences, les regards, le visage, etc. Les phénomènes que l’on observe et que l’on enregistre dans une séance de travail en classe fusionnent les processus propres à ces deux dimensions. L’analyse et l’interprétation de ces phénomènes exigent donc que l’on puisse les distinguer. Car, par exemple, le langage peut être mobilisé pour remplir la fonction d’objectivation ou celle de traitement mathématique, selon les tâches d’apprentissage proposées, et celle de communication dans la gestion de chacune des tâches. La compréhension des productions verbales, écrites ou orales peut alors devenir équivoque, si l’on oublie cette multifonctionnalité de la langue naturelle.

Il y a enfin le point de vue des élèves. Il concerne la compréhension de la manière de travailler en mathématiques qui va contre la manière de travailler dans les autres disciplines, comme les sciences de la vie et de la terre, la géographie etc., Les objets mathématiques n’étant pas perceptivement ou instrumentalement accessibles, comme dans ces disciplines, la manière mathématique de travailler exige des manières de voir, de décrire ce que l’on voit, de raisonner, qui vont contre celles spontanément et normalement pratiquées en dehors des mathématiques. Autrement dit, dans les autres disciplines, il n'y a pas à distinguer une face cachée et une face exposée de l'activité scientifique. La manière mathématique de travailler, qui constitue la face cachée de l'activité mathématique, apparaît alors d'autant plus insaisissable que les élèves doivent, *dans la même journée, d'une heure à l'autre,* aussi travailler dans les autres disciplines.

5.3. La face cachée de l'activité mathématique est-elle prise en compte dans l'enseignement des mathématiques et dans les recherches sur l'enseignement?

Il est essentiel non seulement de séparer ces différents points de vue, mais aussi de voir leurs relations.

La distinction entre face exposée et face cachée de l'activité mathématique porte sur la nature de l'activité mathématique. Elle est indépendante de toute considération d'enseignement. Mais les points de vue, mathématique et cognitif, dont elle relève divergent radicalement. Car, lorsqu'on regarde la face exposée des mathématiques, on se focalise sur les concepts (propriétés) sur les procédures (algorithmes et méthodes) qui sont utilisables pour résoudre des problèmes mathématiques ou des problèmes de la réalité. Mais lorsqu'on on regarde la face cachée, on se focalise les gestes intellectuels propres à la manière mathématique de travailler. *Or ces gestes sont transversaux à tous les concepts et à toutes les procédures mathématiques.* La "théorie" des registres porte sur la face cachée de l'activité mathématique. Elle décrit les processus et les facteurs cognitifs qui permettent de favoriser la compréhension en mathématiques. On voit donc que le point de vue cognitif ne peut pas être subordonné au point de vue mathématique. Il est aussi important que le point de vue mathématique, du moins pour l'apprentissage des mathématiques jusqu'à 16 ans.

La distinction entre l'organisation institutionnelle et l'organisation pédagogique de l'enseignement porte sur l'échelle de temps et sur la nature des objectifs. A l'échelle d'un cycle de quatre ou cinq ans, la progression suit l'ordre de (re)construction mathématique des connaissances choisies comme objectifs globaux d'acquisition. A l'échelle d'une ou plusieurs séances en classe, la progression vise l'introduction de chacun des concepts qui ont été définis, à l'échelle de la progression institutionnelle, comme des objectifs locaux et successifs d'acquisition. L'organisation pédagogique reste donc entièrement subordonnée à l'organisation institutionnelle. Il y a ce que l'on doit supposer avoir été acquis dans les classes précédentes et ce qu'il reste à faire "avant la fin de l'année". Cette totale inclusion de l'introduction successive de nouveaux concepts dans une progression institutionnelle conduit donc à ne prendre en compte que la face exposée des mathématiques. Et comme l'enseignement des mathématiques ne peut pas se faire seulement du point de vue mathématique, on cherche un schéma cognitif général d'apprentissage (empirique, pragmatique, génétique, iconique, etc.) qui puisse être utilisé pour introduire chaque nouveau concept.

Enfin, il y a l'écart entre le point de vue des élèves et celui de l'enseignant sur le travail en classe. Cet écart tient au développement personnel de chaque élève. Il ne faut donc pas regarder le développement de la capacité d'initiative et de contrôle, dans la résolution de problème, sur les seules séances d'introduction d'un nouveau concept, mais sur toute l'année et tout au long d'un cycle. Ni les acquisitions réelles des élèves, ni les difficultés profondes de compréhension qu'ils rencontrent ne peuvent être observées à l'échelle locale de quelques séances, et encore moins sur une seule séance.

Pour apercevoir la complexité des phénomènes que recouvrent les problèmes de compréhension et d'apprentissage des mathématiques par tous les élèves, il faut croiser tous ces points de vue en fonction de leurs relations (Table 4.3). On voit alors que la face cachée n'est pas prise en compte dans l'enseignement des mathématiques, ni au niveau d'une progression institutionnellement organisée, ni à celui de l'organisation du travail en classe.

On n'analyse pas les progrès et les blocages des élèves de la même manière selon ces deux point de vue opposés, c'est à dire selon que l'on considère que les processus de compréhension et d'apprentissage en mathématiques sont les mêmes ou, au contraire, ne sont pas les mêmes que ceux mobilisés dans les autres domaines de la connaissance. *L'enseignement des mathématiques, aussi bien à l'échelle de la progression institutionnelle qu'à celle de l'organisation pédagogique du travail en classe, s'en tient à la face exposée des mathématiques.* On peut dire la même chose de presque toutes les recherches sur l'enseignement des mathématiques.

	La face EXPOSEE **de l'activité mathématique :** LES CONCEPTS ET PROCEDURES relatives à des objets uniquement accessibles à travers des systèmes de représentation sémiotique: *propriétés, algorithmes, démonstrations*	La face CACHEE **de l'activité mathématique :** Les manières de VOIR, d'EXPLORER, de DEFINIR, *EN SAUTANT d'une représentation à l'autre POUR UN MEME OBJET*. Elles sont 1. CONTRAIRES à celles pratiquées en dehors des mathématiques 2. INDEPENDANTES des objets et "concepts" mathématiques enseignés.
ORGANISATION INSTITUTIONNELLE de l'enseignement sur un cycle (primaire, collège)	1 Choix des objectifs globaux de formation mathématique : des connaissances complètes 2. Décompositions en "concepts" prérequis, à acquérir successivement sur plusieurs années.	NON PRISE EN COMPTE *Recours à des théories générales de l'acquisition des connaissances, comme si les processus cognitifs requis pour comprendre en mathématiques étaient les mêmes que pour les autres disciplines.*
ECHELLE DE TEMPS pour l'observation des processus d'apprentissage dans les classes.	1. LES SEANCES EN CLASSE organisées pour l'objectif local d'acquisition d'un concept et *les évaluations locales* au terme d'une séquence 2. *Les enquêtes globales* (nationales et internationales) à la fin d'un cycle.	QUELQUES SECONDES : l'échelle où se joue la reconnaissance de ce qui est représenté PLUSIEURS SEMAINES : *séquences d'activités* UNE ANNEE OU PLUS : *suivi longitudinal des élèves*)
ERREURS ET DIFFICULTES prises en compte	Celles liées à un concept ou un algorithme	LESFAUSSES RECONNAISSANCES RECURRENTES (jamais surmontées au long du cursus)
POINT DE VUE PRIVILIEGIE	L'ENSEIGNANT ET LA CLASSE	LES ELEVES face à la manière de travailler en mathématique

Table 4.3 Divergences entre deux approches de l'apprentissage des mathématiques.

6. Conclusion: trois questions préalables au “choix” d’une théorie cognitive

Des réussites mathématiques locales à des questions, ou dans une résolution de problème impliquent-t-elles la compréhension d’un point de vue cognitif?

Cette question porte sur les pratiques d’évaluation et sur l’élaboration des questionnaires d’évaluation en mathématiques, aussi pour les évaluations après une séquence d’apprentissage que pour les enquêtes nationales et internationales. Les “réussites” enregistrées correspondent-elles à une compréhension qui va permettre soit de nouvelles acquisitions soit une utilisation des connaissances acquises dans les multiples situations de réalité où elles peuvent être appliquées? Ou, au contraire, masquent-elles une incompréhension qui va conduire ultérieurement à des échecs et des difficultés croissantes de compréhension? La valeur diagnostique et pronostique des évaluations dépend du critère de compréhension qui a déterminé le choix des tâches pour élaborer un test d’acquisition. Le plus souvent, ce critère est uniquement l’exactitude du résultat obtenu. Car le deuxième critère mathématique s’avère difficile à utiliser, les enseignants et les chercheurs étant souvent réduits à relever dans les productions des élèves, la présence de quelques mots pris comme indicateurs de compréhension. En revanche, le critère cognitif de reconnaissance dans les différents cas de conversion n’est jamais pris en compte.

Comment prendre en compte, dans l’enseignement, la face cachée de l’activité mathématique?

Pour les fonctions, par exemple, on va habituellement de l’écriture algébrique d’une équation au graphe, en faisant tracer ce graphe. Mais cette pratique privilégie une appréhension locale des points d’intersection au détriment d’une appréhension globale des valeurs visuelles du graphe. Et la très grande majorité des élèves en restent à une interprétation iconique des graphes qui est pour eux sans rapport avec la représentation algébrique des fonctions et qui devient vite trompeuse. Leur utilisation des deux registres reste désarticulée. Il n’y a entre les deux registres aucune coordination. La compréhension exige d’abord que les élèves puissent reconnaître l’expression algébrique représentée, quelles que soient les variations possibles du contenu visuel qualitatif des graphes. Autrement dit, le critère cognitif de reconnaissance porte sur un ensemble de représentations différentes possibles, et non pas sur les quelques situations typiques étudiées en classe en relation avec le concept de fonction. Cela exige l’élaboration de tâches spécifiques de reconnaissance visant faire discriminer les correspondances entres les variables qualitatives des graphiques et les unités de sens constituant les expressions algébriques.

Ces tâches sont des tâches propédeutiques. Elles doivent être préalables à l'introduction des concepts de fonction ou de surface quadratique, etc. Car elles concernent les gestes intellectuels qui permettent de comprendre. Le seul fait de les évoquer soulève souvent des réticences, ou suscite des rejets de principe, car avec ce type de tâches "on ne fait plus de mathématiques" ou on détournerait l'attention des objets mathématiques sur leurs représentations intermédiaires. Mais de telles objections nous renvoient aux trois distinctions clés pour analyser la connaissance et aux questions qu'elles soulèvent pour analyser l'activité mathématique.

Quel est l'enjeu d'un enseignement des mathématiques pour tous les élèves avant 15 ou 16 ans?

Tout d'abord, les objectifs globaux de l'enseignement des mathématiques ne peuvent pas être les mêmes, lorsque l'enseignement s'adresse à tous les élèves jusqu'à 15-16 ans et lorsqu' il se diversifie en filières, en fonction d'une préorientation profession professionnelle. L'enjeu de l'enseignement des mathématiques est de donner aux élèves les moyens de "faire des mathématiques" par eux-mêmes c'est à dire d'être capables d'initiative et de contrôle dans la résolution de problèmes et d'apprendre à apprendre les mathématiques qui, selon les orientations professionnelles choisies, leur seront réellement utiles. Dans cette perspective, la face cachée de l'activité mathématique est aussi fondamentale que la face exposée. Ce qui n'est plus le cas pour les filières spécialisées.

En ce sens, l'enjeu fondamental de l'enseignement des mathématiques jusqu'à 16 ans, doit être de développer l'autonomie intellectuelle des élèves. C'est sous cet aspect que les mathématiques peuvent apporter une contribution majeure par rapport aux autres disciplines enseignées, à la formation générale des élèves. Il semble que nous soyons très loin de cet objectif. Rien n'est plus révélateur d'une méprise sur les objectifs et les apports de l'éducation mathématique, et sur son échec, que l'argument que nous avons souvent entendu dans la bouche des enseignants et dans celle des élèves: "cela n'a pas été vu en classe". Cet argument est la négation même du critère psychologique de la réussite d'un apprentissage: le transfert à des situations entièrement nouvelles.

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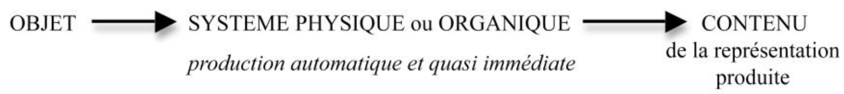
ANNEXE

Représentations non sémiotiques et représentations sémiotiques

La différence entre les représentations sémiotiques et les représentations non sémiotiques tient à l'existence, ou non d'un rapport de causalité entre l'objet et les systèmes producteurs de représentation.

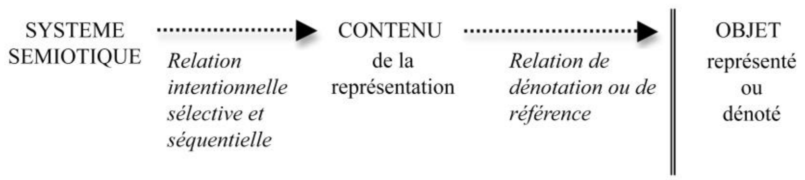
Lorsque les systèmes producteurs sont des systèmes physiques ou organiques nous avons le schéma suivant de production dans lequel *les deux flèches représentent la relation de causalité* :



Les représentations non sémiotiques présentent deux caractéristiques :
— le processus de production de la représentation échappe au contrôle du sujet. Il est non intentionnel
— la relation du contenu de la repésentation à ce qu'elle représente n'est jamais une relation de référence.

Ainsi *le contenu d'un image produite physiquement par un appareil*, c'est à dire sa capacité informative est limitée par le pouvoir de résolution, ou pouvoir séparateur, du capteur utilisé (optique, oeil humain...).

Lorsque les sytèmes producteurs sont des systèmes sémiotiques, le processus de production est radicalement différent. Dans le schéma ci-dessous, la première flèche en pointillé représente le processus selectif, séquentiel et intentionel du processus de production. Et la seconde flèche représente la relation de dénotation ou de référence du contenu à l'objet représenté. La double barre marque le caractère absent ou inaccessible de l'objet représenté.



Les représentations sémiotiques présentent deux caractéristiques :
— le processus de leur production est intentionnel, sélectif et séquentiel.

— la relation du contenu de la représentation à l'objet, absent ou inaccessible est seulement une relation de référence.

Ainsi le contenu d'une image produite sémiotiquement, c'est à dire sa capacité informative la capacité informative dépend du choix des unités figurales intentionnellement sélectionnées, et de leur valeur oppositive à d'autres unités figurales ainsi que de leur combinaison avec d'autres en une organisation spécifique (Duval 2006c, p. 125).

On voit donc le fossé cognitif considérable qui sépare les représentations non sémiotiques et sémiotiques, alors qu'elles sont souvent confondues ou mises sur le même plan lorsqu'on parle par exemple de « représentations iconiques ».

The Epistemological Foundations of the Theory of Objectification

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1. Introduction

The Oxford English Dictionary[1] defines epistemology as "the theory of knowledge and understanding, esp. with regard to its methods, validity, and scope." Epistemology tries indeed to answer the question of *how* things are known. It is clear, therefore, that educational theories cannot go far without resorting to epistemology. In other words, epistemology appears as a prolegomenon to any educational theory.

It is not surprising that epistemology has always been of interest to mathematics educators[2]. Nor is it surprising that the foundational theories in mathematics education — constructivism, and the theory of didactic situations, for instance — resorted to epistemology. Constructivism[3] resorted to Piaget's genetic epistemology in the adapted Kantian version offered by von Glasersfeld (1995) and its idea of *viable knowledge*. The theory of didactic situations[4] resorted to Piaget's genetic epistemology and also to Bachelard's (1986) epistemology and its idea of *epistemological obstacles*.

Since the question of *how* things are known can be answered in different ways, it is not unexpected that epistemology comes in a variety of kinds: rationalist epistemology, empiricist epistemology, pragmatic epistemology, etc.

[1] On line: http://www.oed.com/.
[2] See, e.g., Artigue (1990; 1995); D'Amore (2004); Glaeser (1981); Sfard (1995).
[3] Cobb (1995); Cobb, Yackel & Wood (1992).
[4] Brousseau (1997).

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in Laura Branchetti, *Teaching and Learning Mathematics.*
Some Past and Current Approaches to Mathematics Education, pp. 127-149.

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http://isonomia.uniurb.it/epistemologica

Yet, as Wartofksy points out, "Historicaly (*sic*), epistemology has been ahistorical"[5]. What makes epistemology ahistorical is not an inadvertent inattention to history. Rather, it is a shared common essentialism. Wartofsky continues:

> Epistemologists have sought to fix the universal and necessary conditions of any knowledge whatever, or to establish the essential nature of the human mind. Thus, whether empiricist or rationalist, realist or phenomenalist, traditional epistemologies have shared a common essentialism. What made such epistemologies different were alternative accounts of what are the fixed, essential modes of the acquisition of knowledge, or what are the universal and unchanging structures of the human mind.[6]

Piaget's genetic epistemology is an interesting case in point. In order to understand how we know, Piaget does indeed resort to history[7]. However, the mechanisms of knowledge construction that Piaget identifies in his genetic epistemology (i. e. assimilation, adaptation, equilibration) are *universal*; they do not depend on the geographic or the temporal situation. The mechanisms of knowledge construction are both *a*historical and *a*contextual[8]. As a result, history does not play any epistemological constitutive role (other than as a marker of a naturalist phylogenetic evolution of the species). Hence, if Piaget resorts to history, it is only to refute it. The reasons may be found in a kind of essentialism with which Piaget endows his genetic epistemology. Piaget's essentialism is not of a Platonic nature. It does concern the immutability of the objects of knowledge. The immutability concerns rather the manner in which he interprets human action; that is, as schemas that become organized into fixed logical-mathematical structures. He says: "There is no experimental data that suppose, if only for its reading, a logical-mathematical coordination (of any level, even sensorimotor) to which this data is necessarily relative"[9].

However, the ahistoricity of epistemology is at odds with the main tenets of contemporary sociocultural approaches, in particular those approaches that argue for the cultural situatedness and historical nature of knowledge and knowing[10]. The question is: Is there a possibility for a non-essentialist kind of epistemology? Is there a possibility for thinking of an account of the way in

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[5] Wartofsky (1987: 357).
[6] Wartofsky (1987: 357).
[7] See, e.g. Piaget & Garcia (1989).
[8] Radford, Boero & Vasco (2000).
[9] "Il n'existe pas de donnée expérimentale qui ne suppose, ne fût-ce que pour sa lecture même, une coordination logico-mathématique (de n'importe quel niveau, fût-ce sensori-moteur) à laquelle cette donnée est nécessairement relative". Piaget (1950: 17).
[10] D'Ambrosio (2006); D'Amore, Radford & Bagni (2006).

which we come to know that will really take into account history and culture as epistemic categories?

The answer is yes, and some efforts have been made in the past. A few decades ago, the science epistemologist Marx Wartofsky, in his article *Epistemology Historicized*, offered some ideas about how a historical epistemology would look. Such an epistemology, he suggested, would start "from the premises that the acquisition of knowledge is a fundamental mode of human action"[11]. But instead of considering human action in a formal or abstract way, as Piaget did, he suggested to understand it as a form of human practice inseparable from other forms of human practice, "inseparable from the historicity of these other modes, that is, from their historical change and development"[12]. Such an epistemology should be based on the idea that:

> The appropriate domain for the study of human cognitive practice is not the abstract and relatively featureless domain of the 'human mind', whether tabula rasa, or packed full of innate ideas or faculties; but rather the concrete, many-featured and historical domain of human practices — social, technological, artistic, scientific.[13]

Wartofsky also called attention to the epistemic role of semiotics and artifacts (material objects, symbols, representations) and strongly claimed that since artifacts and symbols have a history, so does cognition: "modes of cognitive practice, perception, thought, ways of seeing and ways of knowing, also have a history"[14]. As a result, "modes of cognition change historically in relation to changes in modes of social practice, and in particular, in relation to historical changes in modes of representational practice"[15].

In the case of mathematics education—the research domain in which I would like to place this discussion—a historical epistemology should be concerned with the elucidation of the nature of objects of knowledge as cultural-historical entities, particularly with their nature and knowability. Such an epistemology should show how the knowability of mathematical objects is cast within definite evolving historical modes of cognition.

The purpose of this article is to offer a discussion of some elements of an epistemological nature that underpin the theory of objectification[16] and to which the theory has recourse in order to conceptualize teaching and learning,

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[11] Wartofsky (1987: 358).
[12] Wartofsky (1987: 358).
[13] Wartofsky (1987: 358).
[14] Wartofsky (1987: 358).
[15] Wartofsky (1987: 358).
[16] Radford (2008; 2013; 2014a).

and knowledge and knowing. The kind of historical epistemology to which the theory of objectification resorts draws on Hegel's work and the dialectic materialist school of thought as developed by Marx (1998) as well as some dialectician philosophers and psychologists after him, such as Evald Ilyenkov (1977), Theodor Adorno (1973, 2008), L. S. Vygotsky (1987), and A. N. Leont'ev (1978). The historical epistemology leads to envision knowing and learning against the background of historical modes of cognition and forms of knowability.

Given the legendary contempt that Hegel showed for mathematics[17], my enterprise, to say the least, is daunting. Hegel[18] was indeed critical of the mathematics of his time. Hegel sensed in a very clear way that mathematics was turning into a technical discipline, which, through its universalist claims and aspirations, sacrifices meaning in the interest of calculations. Hegel, I would say, would have been rather sympathetic to something like a "poetic mathematics," an expressive adventure mediated by an expressive language where subject and object co-inhabit together. In the mathematics of Hegel's time, however, the language of mathematics was already a language without subject. In the mathematics of Hegel's time, the individual had already evaporated from the mathematical discourse. As the German philosopher Theodor Adorno puts it, "The subject is spent and impoverished in its categorial performance; to be able to define and articulate what it confronts . . . the subject must dilute itself to the point of mere universality"[19].

Yet, I will draw on Hegel's dialectics to talk about mathematical objects, knowledge, and knowing. I do think that despite Hegel's well-known idealism and anti-mathematical stance, he provides elements with which to understand knowledge in general and mathematical knowledge in particular.

I shall start by addressing the questions of the nature of mathematical objects and how we think about these objects. It is already a Hegelian insight that thinking and its objects cannot be dealt with separately. To think, indeed, is to think about something. Thinking and this something that is the object of thinking are intertwined and indissoluble.

2. Mathematical objects

There are several widespread approaches to mathematical objects. In this section, I will mention three of them. The first one consists of conceiving of

[17] See, e.g., Hegel (1977; 2009).
[18] As well as other famous philosophers, like Heidegger (1977) and Husserl (1970).
[19] Adorno (2008: 139).

mathematical objects as produced by the mind. This is the approach articulated by Descartes, Leibniz, and other rationalists. In his *New Essays Concerning Human Understanding*, Leibniz says:

> All arithmetic and all geometry are innate, and are in us virtually, so that we can find them there if we consider attentively and set in order what we already have in the mind, without making use of any truth learned through experience or through the tradition of another, as Plato has shown in a dialogue in which he introduces Socrates leading a child to abstract truths by questions alone without giving him any information. We can then make for ourselves these sciences [i.e., arithmetic and geometry] in our study, and even with closed eyes, without learning through sight or even through touch the truths which we need; although it is true that we would not consider the ideas in question if we had never seen or touched anything.[20]

The mind, therefore, has only to search inside itself to tidy up and order out what is already there to find mathematical objects and what can be said about them.

There is a second approach — chronologically older than the previous one — that goes back to Plato. Plato thought of mathematical objects as *forms*: unchanging entities that populated an ideal world of perfect and intelligible atemporal essences[21]. How do we come to know these unchanging forms? There are two widespread answers.

The first answer comes from Plato. In his *Phaedrus* dialogue, Plato explained the knowability of the objects of knowledge in terms of recollection. Our soul was assumed to have been in touch with the realm of forms, the realm of Truth, when the soul "disregarded the things we now call real and lifted up its head to what is truly real instead"[22]. Unfortunately, during our birth in the world, we forgot about Truth and forms. The process of recollection is well illustrated in another dialogue, *Meno*, the one Leibniz was referring to in the previous citation, where a slave is presented as going into a process of reminiscence: he is recollecting knowledge about geometric figures that he already had in a past life.

The second answer is the modern answer: the unchanging forms are *discovered*. In a recent article Côté[23] summarizes the point in the following terms: "the full version of mathematical Platonism means that mathematicians do not invent theorems, but discover them".

[20] Leibniz (1949: 78).
[21] See, e.g., Caveing (1996).
[22] Plato (2012: 235-249c).
[23] Côté (2013: 375).

Platonism remains very popular among mathematicians, at least if we are to believe Bernays (1935) and more recently Giusti (2000). However, today Platonism does not seems to be articulated in terms of recollection; rather, it seems to be articulated in terms of *discovery*. I know of no mathematician or mathematics educator resorting to Plato's reminiscence theory to explain learning. The theory of reminiscence seems to have fallen out of favour. Yet, we cannot say that Plato has not influenced us. For one thing, Piaget drew on Kant, who, in turn, drew on Plato. Kant drew in particular on Plato's ontology and referred to the forms as *noumena*. In Kant's account[24], they are prior to, and independent of, human activity. They are *somewhere* already. They exist. To convey that idea of the independence of these objects from human activity, Kant referred to them as *things-in-themselves*.

Let me turn now to the third account of objects of knowledge — the constructivist account. Although Piaget (1924) drew on Kant, he was quick to remove Kant's aprioristic stance: in Piaget, objects of knowledge are rather the product of the individual's constructions. Mathematics education has largely adopted this sense in order to talk about knowledge.

I would like to highlight that the fundamental metaphor behind the idea of objects of knowledge as something that *you make* or something that *you construct* is that objects of knowledge are somehow similar to the concrete objects of the world. You construct, build, or assemble objects of knowledge, as you construct, build, or assemble chairs. This idea of knowledge as construction is relatively recent. It emerged slowly in the course of the 16th and 17th centuries when manufacturing and the commercial production of things became the main form of human production in Europe. Hanna Arendt summarizes this conception of knowledge as follows: "I 'know' a thing whenever I understand how it has come into being"[25]. It is within the general 16th and 17th centuries' outlook of a manufactured world that knowledge is first conceived of as a form of manufacture as well. When Kant writes at the end of the 18th century his famous *Critique of Pure Reason*, he is articulating and expressing, at the theoretical level, the new cultural view of knowledge — the view of the modern period in Western development. In the 20th century, and with Piaget (1970) and von Glasersfeld (1995) in particular, the individualist dimension of the modern view on knowledge was pushed to its last consequences: You and only you construct your own knowledge. For, in this view, knowledge is not something that someone else can construct and pass on to

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[24] See Kant (2003).

[25] Arendt (1958: 585).

you. Doing, knowing, and learning are conflated. What you *know* is exactly what you have *learned* and *done* by yourself[26].

As many scholars have pointed out, such a view of knowledge is problematic on several counts. For instance, it leaves little room to account for the important role of others and material culture in the way we come to know, leading to a simplified view of cognition, interaction, intersubjectivity, and the ethical dimension. It removes the crucial role of social institutions and the values and tensions they convey, and it de-historicizes knowledge[27].

In the following section I explore the idea of objects of knowledge from a neo-Hegelian perspective.

3. Knowledge from a neo-Hegelian perspective

In this section I outline the Hegelian-Marxist dialectical materialist conception of knowledge that is at the heart of the theory of objectification[28]. Knowledge, in this theory, is not something that individuals possess, acquire, or construct. The conception of knowledge is rather based on a distinction between two related although different ontological categories: *potentiality* and *actuality*. The potentiality/actuality distinction goes back to Aristotle who used the words *dunamis* and *energia*. Potentiality (*dunamis* in Greek) designates the source of motion, something that is entangled in the material world. Potentiality is synonymous with “capacity” or “ability” or “power.” Living things and artifacts—musical instruments, for example — have potentiality; that is, a definite capacity *for doing something*. Aristotle contrasted this potentiality to *actuality*, which is “being-at-work” — something in motion occurring in front of us.

Objects of knowledge (mathematical and other) belong precisely to the category of potentiality, and as such, are *abstract* or *general*; that is, they are conceived of as “undeveloped, lacking in connections with other things, poor in content, formal”[29].

Objects of knowledge are hence not psychological or mental entities. They are *pure possibility* — a “complete totality of possible interpretations — those already known, and those yet to be invented”[30]. They are possibility

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[26] Radford (2014b).
[27] See, e.g., Campbell (2002); Lerman (1996); Otte (1998); Roth (2011); Valero (2004); Zevenbergen (1996).
[28] Radford (2008).
[29] Blunden (2009: 44).
[30] Ilyenkov (2012: 150).

grounding interpretations and actions; for example, possibility of making calculations, or thinking and classifying spatial forms in certain "geometric" manners, or possibilities of imagining new ways of doing things, etc.

Objects of knowledge as possibility are not something eternal and independent of all human experience (like Kant's idea of things-in-themselves or as Plato's concept of forms). Objects of knowledge are social-historical-cultural entities. In fact, they result from, and are produced through, social labour. In more precise terms, objects of knowledge are an evolving culturally and historically codified *synthesis* of doing, thinking, and relating to others and the world.

Let me give you an example. It comes not from humans but from chimpanzees. As we know, some groups of chimpanzees crack nuts. Primatologists have shown the complexity underneath the actions that leads to cracking a nut: the chimp has to make several choices. First, the chimp has to choose the nut; second, the chimp has to choose the first stone where the nut will sit (the anvil stone); third, the chimp has to choose the hammer stone, then choose the precise pressure with which to apply the hammer stone to the nut so that the nut is neither crushed nor left unopened. Matsuzawa, Biro, Humle, Inoue-Nakamura, Tonooka, and Yamakoshi (2001: 570) show a picture in which Yo, a member of a chimp community in the southeastern corner of the Republic of Guinea, cracks a nut while two young chimps watch her attentively. Yo and other chimps' sequence of actions for cracking a nut becomes a historically codified *synthesis*, resulting in an object of knowledge—knowing how to crack nuts. We can qualify this object of knowledge as *kinesthetic* in that it involves bodily actions without language and signs. In general, in the case of human objects of knowledge, in addition to artifacts, the actions codified in a cultural synthesis include language and other semiotic systems (diagrams, for instance), providing the resulting object of knowledge with a complexity that may surpass the one found in chimps and other species. The synthesis is often *expressed* in a semiotic system, providing the object of knowledge with a description or definition (a circle *is* …), although the explicit expression of the synthesis is not a condition for the existence of the object of knowledge. How much explicit expression is required will depend on the cultural form of mathematical reasoning that embeds the objects of knowledge. To give a short example, Babylonian scribes were not inclined to provide specific definitions of the objects they dealt with (e.g. circles, squares, rectangles, etc.). They talked about objects without defining them. We find the exact opposite in Greek mathematics. Thus, in Euclid's *Elements*, there is an obsessive need to define the mathematical objects from the outset.

I will come back to this point later. For the time being, let me say something about the idea of *synthesis* that is central to the definition of objects of knowledge that I have just suggested. The synthesis to which I am referring is not the kind of synthesis Kant[31] put forward in the *Critique of pure reason*; that is, the synthesis of a legislative reason that merges and subsumes an array of cases into a (pre-) given concept. The materialist concept of synthesis that I am suggesting here conceives of synthesis as *codified labour*. More precisely, it refers to a nascent relationship of various and different actions out of which these actions become recognized as *different* and the *same*. In the chimps' example, a synthesis of actions that have been carried out by *different* chimps with *different* stones and *different* nuts in *different* moments become recognized as *same-yet-different*. The synthesis that leads to the object of knowledge makes this object general in the sense that it does not relate to this or that situation (these stones and nuts, Yo, or another particular chimp): It is a *synthesis* of different singularities and as such it is *not* an abstraction, but a synthesis that contains the divergence and contradictions of the singularities that it attempts to hold together. It is a *synthesis of non-identity*, which confers the object of knowledge with its *internal contradictions*. Instead of being a flaw or imperfection, the non-identitarian synthesis confers the object of knowledge with an irreconcilable nature vis-à-vis the synthesized items. The resulting and unavoidably internal contradictions are precisely what afford the further development of the object of knowledge. As bearer of contradictions, the object of knowledge indeed opens up room for further actions and new interpretations and creations. As such, the object of knowledge always points to what itself is not.

To come back to the chimpanzee example, some chimps crack nuts of certain kinds, but not others. The nut-cracking object of knowledge may eventually evolve if the chimps start including other nuts or hitting nuts with objects other than stones (which has happened in some chimp cultures where tree branches started being considered). Initially, Yo's community cracked coula nuts, but not panda nuts. After some time, they started cracking panda nuts as well. A new synthesis occurred. The new synthesis *sublated* the previous one, giving rise to a development of the object of knowledge.

The same may be said of mathematical objects of knowledge. They are cultural and historical synthesis, of dealing with, for instance, certain kind of situations or problems that mathematicians solve through a sequence of well identified steps. The situations become expressed as, for example, "linear

[31] Kant (2003).

equations." Fig. 1 shows a linear equation and a sequence of well identified steps to solve it by a group of 11–12-year-old Grade 6 students.

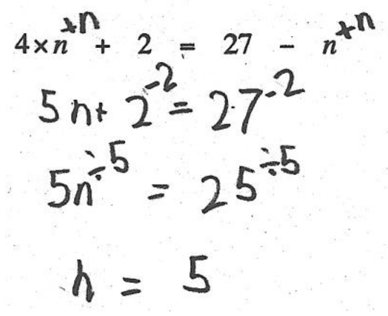

Figure 1. A group of Grade 6 students' sequence of steps to solve a linear equation

Let me note, however, that linear equation knowledge is not the sequence of signs we see on the paper. Linear equation knowledge is a synthesis, a codified way of dealing with problems or situations like the one shown in Fig. 1. Linear equation knowledge is pure cultural possibility — possibility of thinking about indeterminate and known numbers in certain historically constituted analytical ways. In the example referred to in Fig. 1, linear equation knowledge has been *realized* or *actualized* in a singular instance, the solving of the equation 4xn + 2 = 27 — n. This actualization of the linear equation knowledge is what, following Hegel's dialectics, may be termed a *singular*. So, from potentiality, linear equation knowledge has been put in motion and passed from something general (something undeveloped and poor in content) to something concrete and actual; something noticeable and tangible, in short, a singular. This singular *is not* the symbols themselves shown in Fig. 1, but the embodied, symbolic, and discursive actions and thoughts required in solving the specific equation 4xn + 2 = 27 — n. In the singular, mathematical knowledge *appears* as both concrete *and* abstract. It cannot be concrete only; nor can it be abstract only. It is both *simultaneously*.

But what is it that makes possible the movement from potentiality to actuality, or from the general to the singular? The answer is *activity* or *labour*[32]. Indeed, knowledge as general, as synthesis, is not an object of thought and interpretation. It lacks *determinations.* The lack of *determinations* renders knowledge impossible to be sensed, perceived, and reflected upon. It can only become an object of thought and interpretation *through specific problem-posing and problem-solving activities*.

[32] Radford (2014a).

What this means is that objects of knowledge cannot be accessed directly. They are not immediate objects. They are *mediated*. They are *mediated by activity*[33]. So, when we talk about knowledge and its objects, we have three elements: the objects of knowledge themselves, their actualization in the concrete world (singulars), and the activity that mediates them (see Fig. 2).

By moving from potentiality into the actual realm of the sensuous, the sensible, and the perceptible (something called in dialectical materialism the *ascent from the abstract to the concrete*), objects of knowledge appear instantiated in singulars.

This movement or ascent from the abstract to the concrete should not be interpreted as a repetition in the mechanical sense of technology. To do so would amount to a complete misunderstanding of the dialectical materialist view of objects of knowledge as *empty forms of difference*, or, as Deleuze[34] says, as "invariable forms of variation." It would also amount to conflating two different layers of ontology: potentiality and actuality. This is what Fig. 2 tries to clarify.

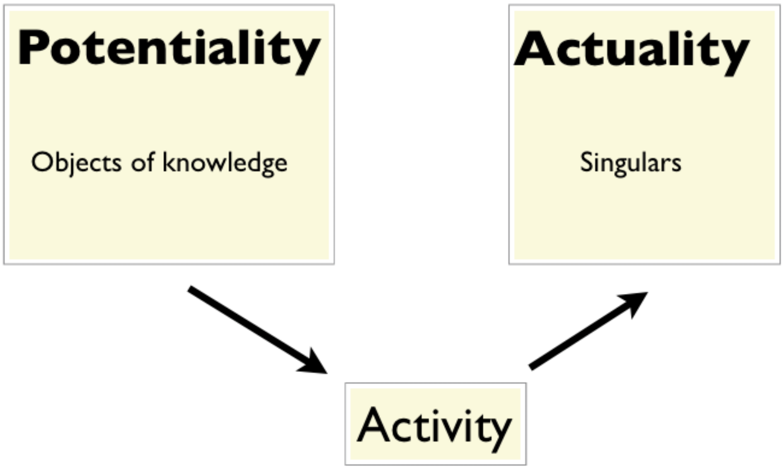

Figure 2. Singulars as the actualization of objects of knowledge through a mediating activity

[33] In opposition to other approaches, in the dialectic materialist epistemological approach I am outlining here, the access to the objects of knowledge is not ensured by signs, but by the individuals' activity. The activity does involve signs, but it is not the signs that reveal the object of knowledge; it is rather the activity.
[34] Deleuze (1968: 8).

4. Concepts

From the individual's viewpoint, what emerges from the actualization or concretion of objects of knowledge through their singulars is the appearance of the object of knowledge in the individual's consciousness. In Hegel's terms, it is a *concept*. In this sense, the concept is a *production,* which means, etymologically speaking, the "bringing forth" or "coming into being" of something. It is in this sense that we refer to classroom mathematical knowledge here: something — an object of knowledge — that comes into being in a singular through classroom activity. The concept is the appearance of the object of knowledge in the student's consciousness through the singular, as afforded by the mediating activity.

Given the crucial mediating nature of the activity in concept formation, we should emphasize here the importance of the classroom activity. If the classroom activity is not socially and mathematically interesting, the ensuing concept and conceptualization will not be very strong. There is hence a pedagogical need to offer activities to the students that involve both the possibility of strong interactional participation and deep mathematical reflections[35].

Naturally, the historically codified way of cracking nuts or dealing with linear equations is not something that chimps in the first case, and students in the second case, grasp directly. It is at this point that we need to consider the concept of *learning*.

In the following section I describe this concept from the viewpoint of the dialectic materialist theory of objectification.

5. Learning

For the young chimpanzees that happen to live in a nut-cracking chimpanzee culture, like Yo's culture, the nut-cracking object of knowledge is pure possibility. The young chimps have to *learn* how to do it. As summarized previously[36], studies in the wild suggest that it takes from 3 to 7 years for the infant chimp to learn the nut-cracking process. Infants do not necessarily start by using a hammer stone and the anvil. The proper attention to the objects, their choice (size, hardness, etc.), and subsequently the spatial and temporal coordination of the three of them (nuts, anvil, and hammer), is a long process. Often, young chimps of about 0.5 years manipulate only one object (either a nut or a stone). They may choose a nut and step on it. As chimps grow older,

[35] Radford (2014b).
[36] Radford (2013).

they may resort to the three objects, but not in the correct sequence of nut-cracking behaviour, resulting in failed attempts. A key aspect of the process is the appearance of suitable cracking skills — for example, "the action of hitting as a means to apply sufficient pressure to a nut shell to break it"[37].

How do chimps learn? They learn by participating in the activity that realizes the object of knowledge. They learn by what Matsuzawa *et al*. (2001) call the "master-apprenticeship" method: They observe, then they try.

We can formulate the question of learning as follows. As *general*, objects of knowledge (i.e. culturally codified ways of doing and thinking) are not graspable or noticeable. In order for an object of knowledge to become an object of thought and consciousness, it has to be set in motion. It has to acquire cultural determinations; that is, it has to acquire content and connections in a process of contrast with other things, thereby becoming more and more concrete. And the only manner by which concepts can acquire cultural determinations is through specific *activities* (in our previous chimpanzee example, through "master-apprenticeship" activities). Learning emerges from the sensuous and conceptual awareness that results from the realization of the object of knowledge (e.g., cracking nuts, solving linear equations) in its concrete realization or individualization.

Let us notice that we always grasp objects of knowledge through the singular that instantiates it; that is, through its individual realization. This is the paradox of learning: in learning we deal with singularities (we solve specific equations, like $4xn + 2 = 27 — n$ or $3xn +5 = n + 9$, etc.). Yet, what we are after is none of those or any other specific equation. We are after culturally constituted *ways of doing and thinking* that can only be grasped obliquely, in an indirect manner, through our participation in the activity that makes this way of thinking present in the singular.

I can now formulate the concept of objectification through which we thematize learning. Objectification is this social co-transformative, sensuous sense-making process through which the students gradually become critically acquainted with historically constituted cultural meanings and forms of thinking and action. Those systems of thinking (algebraic thinking or statistical thinking, for instance) are there, as potentiality for the novice students. When the students cross the threshold of the school for the first time, the objects of knowledge are pure open potentiality. It is through processes of objectification embedded in the activity that mediates the potential and the actual, and

[37] Hirata, Morimura & Houki (2009: 98).

through their realization in concepts, that these systems of thinking will become objects of consciousness and thought.

To simplify our terminology, let us refer to the knowledge of a culture as the ensemble of cultural ways of doing and thinking. Because of the continuous transformation of these ways of doing and thinking, which “change historically in relation to changes in modes of social practice”[38] and the social production of the individual’s existence, the knowledge of a culture is a flexible and dynamic *system*. This system, along with the ensuing ensemble of realizable mediating activities through which knowledge can be actualized, set the parameters of historical modes of cognition and forms of knowability. These modes of cognition and forms of knowability frame, in turn, the scope of concepts that can be produced at a specific time in a specific culture. These concepts also constitute a dynamic system, which, to distinguish from Knowledge, we term Knowing. We then get the diagram shown in Fig. 3.

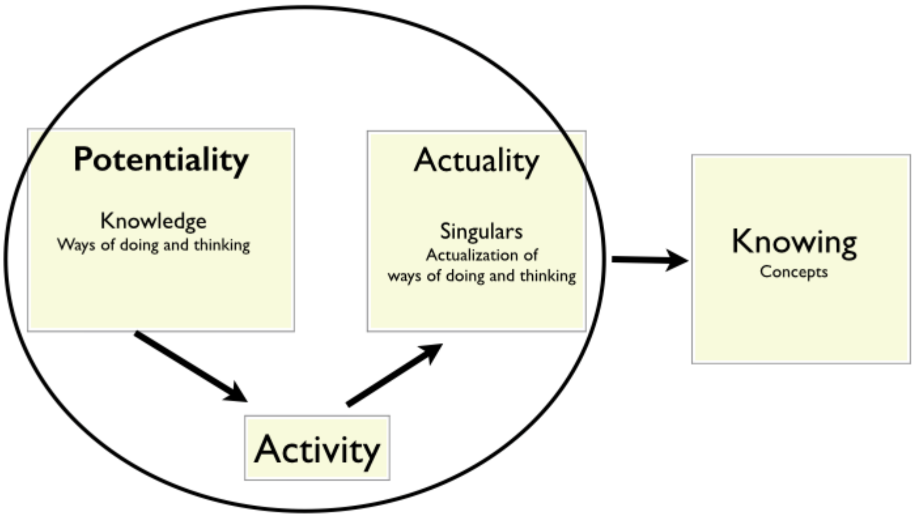

Figure 3. Knowing as what is grasped by the individuals in the realization or actualization of knowledge through activity.

Let me come back once more to the algebra example. Algebra includes several *themes* (generalization, linear equations, polynomial equations, abstract algebra, etc.). A specific culturally codified way of thinking and doing is associated with each one of these *themes.* For example, a linear equation way of thinking and doing comprises ways of posing, reasoning, solving, and

[38] Wartofsky (1987: 358).

dealing with situations susceptible to being expressed in what is meant by linear equations in a certain period and in a certain culture (linear equations may refer to equations involving specific semiotic systems, some type of coefficients only, e.g., positive integers, etc.). Through their cultural codified synthesis, these ways of thinking and doing, which have emerged out of practical cultural activities, may be different from one culture to another (we can think, for instance, of the Ancient Chinese linear equations or those that we find in the Ancient Greek tradition). These ways of thinking and doing (Knowledge in our terminology) have been refined in the course of long cultural processes, which often involve societal contradictions (as it will become clearer in the next section). They become potentiality. Fig. 3 suggests that the activity provides them with particular determinations, depending on the nature of the activity. Thus, the knowledge that is actualized in solving the equation shown in Fig. 1 involves forms of thinking that include positive and negative coefficients, but not fractional or irrational coefficients. What is therefore actualized is something specific. This is the singular: the actual appearance of the general. Now, the concept is what the students grasp of this singular. In other words, the concept is constituted from what has actually become the object of consciousness for the students in the course of their joint labour with the teachers — the sensuous and actual way of thinking and doing *as encountered* and *cognized* by the students. Concepts, of course, are not isolated entities; they also constitute systems — which we have termed Knowing.

6. The politics of knowledge

The idea of knowledge as an ensemble of ways of thinking and doing as developed in the previous section — that is, as culturally codified syntheses of people's actions — allows us to grasp their cultural and historical nature. We have already contrasted the Euclidean insistence in defining things with the Babylonian mathematical thinking where definitions are not required. Euclid worked within the aristocratic Athenian tradition that confers to language an epistemic value that we do not find in the practical approach of the bureaucratic structure of Mesopotamian cities. From the dialectic viewpoint outlined here, ways of thinking and doing are evolving entities rooted in, and informing, practical activity. They arise as the synthesizing effect of activity and, in turn, affect activity (and the individuals who participate in the activity). They are cause and effect, although not in a causal manner. Rather, they are simultaneously cause and effect in a dialectical sense.

To understand a specific way of thinking and doing, we have to turn to the culture in which they operate and to ask about the activities in which people engage. But a real understanding of a specific way of thinking and doing requires, because of the developmental nature of it, looking at it *historically*. As Ilyenkov[39] put it, “A concrete understanding of reality cannot be attained without a historical approach to it.” The developmental nature of knowledge is based on the internal contradictions knowledge bears within itself — contradictions that appear and reappear in the confrontation of objects of knowledge in concrete activities, where they give rise to concepts. These internal contradictions are not logical flaws, nor are they merely epistemological contradictions between opposing or competing purified entities. They are replete of social and societal contradictions. In fact, the internal contradictions of objects of knowledge reflect the societal contradictions from where they emerge. There are questions of social, cultural, and political legitimacy that are brought to the fore that favour some ways of doing to the detriment of others[40].

Let me finish with an example. The Italian mathematician Rafael Bombelli wrote a famous treatise in the 16th century — *L’algebra*. Commenting on Bombelli’s goal, Jayawardene notes:

> Whereas the works of his [Bombelli] predecessors contained many problems of applied arithmetic (some of them solved by means of the methods of algebra), Bombelli’s Algebra contained none. His were all abstract problems. In fact, in the introduction to Book III he said that he had deviated from the practice of the majority of contemporary authors of arithmetics who stated their problems in the “guise of human actions”: “sotto velame di attioni, e negotij humani ... (come di vendite, compere, restitutioni per- mute; cambij; interessi; deffalcationi, leghe di monete, di metalli; pesi; compagnie, e con perdita, e guadagno, giochi, e simili altre infinite attioni, e operationi humane).” He said that these men wrote with a different purpose-they were practical rather than scientific-and that he, on the other hand, had the intention of teaching the higher arithmetic (or algebra) in the manner of the ancients.[41]

Bombelli’s decision has to do with the social competition among cities for work and prestige in 16th century Italy. Bombelli had recourse to the Renaissance praised value on the ancient Greek mathematics. And after having included practical problems in his manuscript “in the guise of human actions,”

[39] Ilyenkov (1982: 212).
[40] See, e.g., Shapin (1995).
[41] Jayawardene (1973: 511).

he drew on Diophantus's work and removed the practical problems. *L'algebra* was made more "scientific" and was directed not the merchants or the abacists but to the aristocratic audience of scientific thinkers of his time[42].

In Bombelli's *L'algebra* we find a neat synthesis of contrasting and contradictory views that are synthesized along the lines of the societal conflicts that ended up in the creation of one of the most elaborate symbolic systems to deal with Renaissance algebra, although abandoned later for other symbolic systems. Bombelli's work shows the sublation of commercial algebra and its development into a scientific version, while showing at the same time that development of objects of knowledge are, as cultural synthesis, cultural and political.

What are the implications of the outlined cultural-historical, dialectic materialist approach to knowledge and knowing? I would like to mention two. First, by considering objects of knowledge as syntheses of people's labour — syntheses that present themselves as potential sources of new interpretations and actions — we move away from situationist, distributionist, individualist, and interactionist accounts of knowledge formation that are at odds to account for the historicity of knowledge and their cultural nature. Second, the dialectic materialist approach emphasizes *the role of activity* in producing knowledge. In doing so, we can reconceptualize knowledge not as the subjective deeds of individuals, but as something that emerges from, and attempts to respond to, problems of a societal nature. Objects of knowledge are bearers of contradictions. These contradictions are not the result of imperfections. They reflect the variety of the individuals' perspectives, interests, and needs (practical, but also aesthetic, ethical, and others) that we find in a culture. They reflect also the manner in which power is distributed in a culture. As a result of these contradictions, they remain open to be expanded, transformed, or refuted in practice.

My account of knowledge and knowing suffers, though, from a lack of attention to the individuals who are in the process of knowing. In fact, this is the inadequacy of the formulation of the problem of *subject* and *object* in traditional epistemology. The subject appears as already given, and invariable. The problem is posed as if the knower is already there, fully constituted or constituted through his or her own deeds. What is inadequate in this way of posing the problem of the relationship between subject and object is that it misses the fact that there is a dialectical relationship between knowing and becoming. We are knowing because we are becoming. And we are becoming because we are knowing (Radford, in press). In the same way that cultures

[42] For the social context see, e.g., Bernardino (1999); Biagioli (1993); Hadden (1994).

offer ways of thinking and doing, they offer ways of becoming. It is my hope that we will soon start exploring this dialectic between knowing and becoming in more systematic ways.

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Saber, conocer, labor en didáctica de la matemática: una contribución a la teoría de la objetivación

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1. Premisa

Las contribuciones a la didáctica de la matemática aportadas por Luís Radford y por sus alumnos y colaboradores en los últimos 10-15 años han propiciado en el área de Didáctica de la Matemática una reflexión profunda sobre algunos aspectos que en muchas investigaciones anteriores no se problematizaban o se consideraban ya adquiridos o, en algunos casos, se ignoraban completemente[1]; estos estudios han llevado incluso a discusiones relevantes sobre algunos temas clásicos de la investigación en didáctica de la matemática[2].

La posibilidad de leer/estudiar muchos de sus recientes trabajos, de participar en encuentros, simposios y debates, de seguir tesis de doctorado sobre este tipo de temáticas, me han llevado a algunas reflexiones que deseo hacer públicas mediante este texto. Se trata, sobre todo, de examinar algunas palabras claves de la teoría de la objetivación desde un punto de vista filosófico y sociológico, con el fin de contribuir al debate internacional en curso sobre dicha teoría, incluso aprovechando, cosa que hago esporádicamente y sin explicitarlos, estudios personales de grado y de posgrado en disciplinas diferentes de la matemática y de la didáctica, sobre todo de filosofía y de pedagogía.

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[1] Me limito a tomar como referencia: Radford (1997; 2002; 2003; 2004; 2005; 2006; 2013a).
[2] D'Amore, Radford & Bagni (2007).

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La razón que me llevó al actual estudio se presentó después de haber participado, del 7 al 10 de enero de 2014, en un simposio virtual donde se debatió sobre investigaciones realizadas utilizando como enfoque teórico la teoría de la objetivación. Cada vez que escuchaba las diferentes intervenciones, sentía la necesidad imperiosa de profundizar de forma crítica en algunos de los temas tratados, no tanto por motivos de simplificación conceptual (¡por el contrario!) sino por la necesidad de evidenciar las complejidades que se escondían; y también con el objetivo de reducir, limar y disminuir ciertas aparentes disonancias con otras teorías y otras interpretaciones, intentando buscar raíces comunes.

2. Saber y ontología

Se trata de una de las palabras de mayor uso en didáctica, en todas las didácticas. "El estudiante adquiere *saber*", "Cómo se construye el *saber*", "Cómo se transmite el *saber*", … son temas que han visto interpretaciones diversas por parte de célebres autores que no vale la pena recordar aquí de forma pedante.

He encontrado siempre fascinante el hecho que la raíz lingüística europea de "saber", *sap*, reúne tanto el tener sabor como el tener sabiduría o sentido, compartida también por otras raíces lingüísticas. El tener sabiduría o sentido (algunos hablan de conocimiento) y el tener sabor son el origen ancestral de aquella idea tan sofisticada que hoy se llama *saber*; este hecho exalta el sentido mismo que está en el origen de las palabras: el sentido del gusto se vincula a la capacidad de conocer y de distinguir entre los conocimientos para elegir aquellos adaptados a las situaciones (que es el sentido o que expresa el tener sabiduría).

Pero el saber lleva inevitablemente al estudio preliminar del ser como tal y a reflexionar, por tanto lleva a la ontología.

Se lee en un muy bien conocido fragmento de Parménides[3]: el ser es eterno; porque, si no fuese eterno, existiría alguna cosa antes del ser, lo cual es contradictorio.

En esta consideración se condensa media historia de la filosofía. El discurso sobre el ser lleva a la disputa: "mundo eterno *versus* mundo creado" y al tema "Dios". Pero Dios está dentro del ser, dando razón a Baruch Espinoza; o, ¿cómo distinguir un ser-que-es-Dios de un ser-que-no-es-Dios? Todas las pruebas de la existencia de Dios tienen una cierta coherencia lógica, pero tienen también un evidente *non sequitur*; por ejemplo, pasando de

[3] *Sobre la naturaleza*, fr. 2, vv 3-5; Diels & Kranz (1903-1952).

movimiento en movimiento se encuentra un movimiento, no un ser divino. Así, pasando de causa en causa, etcétera. Por ejemplo Tomás[4] concluye, de su famosa cadena del ser, que existe alguien omnipotente omnisciente etc. *a priori*; conclusión gratuita, con una peligrosa asunción implícita: que la forma de vida más elevada posible sea el ser-persona. Esto coincide con nuestra experiencia: las plantas están más adaptadas a la vida, una secoya vive mil años y algo más, pero la percibimos como un ser inferior a nosotros. En conclusión, la forma de vida suprema tiene intelecto, consciencia, voluntad… Y así nosotros volcamos esta conclusión sobre nuestra idea de Dios.

¿Pero, es necesario que sea así? ¿No seremos nosotros los gusanos planos, que no perciben la tercera dimensión? ¿Somos y sabemos, o sólo somos?

¿Será posible saber sin ser?

Otra pregunta fundamental, expresada exactamente con las palabras de Leibniz: *¿pourquoi il y a plus tôt quelque chose que rien?*[5]; es decir la pregunta metafísica fundamental: *¿cur est aliquid?* Es la pregunta de todo el saber, para responder a la cual nos ayuda nuevamente Parménides, en su profundo esquematismo: porque la nada, siendo precisamente tal, no puede ser. Y volvemos así a: «el ser es, el no ser no es», que aparece ahora mucho menos obvia. La nada no puede darse sin convertirse en algo.

Ser, saber, conocer constituyen por tanto una sucesión causal. Nosotros, los docentes, nuestros alumnos, somos, sabemos, conocemos en forma indistinguible.

3. Conocer y gnoseología

Hecha una rápida referencia al problema ontológico, quiero concentrarme ahora en el aspecto gnoseológico, o del conocimiento. Inútil decir que entre los dos existe una correlación muy fuerte, sería como decir que son homocigotos; pero siempre dos y no uno.

Parto de algunas tesis obvias.

La primera contraposición de base, del tipo de aquella fundamental entre protozoos y metazoos de Linneo, es entre, por así decirlo, los dogmáticos y los escépticos, es decir entre quienes consideran que el conocimiento pueda darse, y quienes lo niegan (obviamente el término "dogmático" debe ser considerado en una particular acepción que emergerá en el contexto de este

[4] Tomás de Aquino, *Summa Theologiae*, I, q (1990).
[5] Leibniz (1714).

mismo trabajo). El conocimiento es y puede ser adquirido; el conocimiento es, pero queda fuera de nosotros.

Ejemplifiquémoslo haciendo referencia a la filosofía antigua.

Ejemplos del primer tipo: Sócrates, Platón, Aristóteles (es decir la línea “ganadora”); del segundo tipo: Protágoras, Gorgia de Lentini, Pirro de Epiro.

El argumento principal de los primeros contra los segundos es: pero ¿si el conocimiento no existe, tú cómo haces para saberlo?, es decir un uso filosóficamente interesante de la *consequentia mirabilis*[6]. Este método deductivo aparece en un famoso paso del *Teeteto*[7] y en la *Metafísica* de Aristóteles[8]; lo cual se puede expresar como sigue: si el escéptico niega todo, incluso el significado, entonces no está diciendo nada que tenga sentido, por tanto él *phytòs estì*, es como una planta, vegeta en un estado pre-humano.

El argumento típico de los segundos está articulado en dos partes: negación del conocimiento racional (lo cual lleva a paradojas, antinomias y sofismas muy bien conocidos), y negación del conocimiento sensible (los sentidos nos engañan, los ejemplos son numerosos a partir de las paradojas de Zenón)[9].

Una vez aceptada la primera tesis, es decir que el conocimiento se pueda dar, la pregunta siguiente es sobre el “cómo”, y aquí las soluciones son diversas, interesantísimas para nuestros estudios específicos en didáctica de la matemática. Intento proceder de forma taxonómica, siguiendo los esquemas clásicos.

Si se da, ¿se da a *priori* o a *posteriori*?

En el primer caso tenemos el innatismo, por ejemplo el anamnesis de Platón[10], para citar el caso típico; en el segundo tenemos el empirismo, el *nihil est in intellectu quod prius non fuerit in sensu*, axioma de la filosofía

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[6] La *consequentia mirabilis* es la formulación de un especial principio de demostración por absurdo que es llamado también “principio de Clavio” por Jan Łukasiewicz (1970), cuya formulación es generalmente dada de la siguiente forma: $(\neg T \rightarrow T) \rightarrow T$. En realidad esa debe ser expresada metalingüísticamente para formular una interesante deducción (usada por Gerolamo Saccheri y Georg Cantor, entre otros), que es la que la hizo célebre: si de $\neg T$ se puede deducir T, entonces T. Hay quienes quiso ver esta deducción ya en Aristóteles, otros en Platón, otros en Sexto Empírico. Una detallada disertación histórico — lógico — analítica relativa a esta deducción se encuentra en D’Amore y Matteuzzi (1972).

[7] Vailati (1911).

[8] En la cual la *consequentia* en mi opinión no es explícita sino sobrentendida, al contrario de lo que se afirma en Łukasiewicz (1951).

[9] Las interpretaciones de la paradoja de Zenón pueden darse a favor o en contra de los Pitagóricos; las dos versiones, aún siendo antitéticas, son ambas admisibles: véase D’Amore (2001a).

[10] Bien conocido el fenómeno de reminiscencia descrito en el *Menón*, en *Fedro* y en otros *Diálogos*; véase Platón (1997).

escolástica, hecho propio hasta convertirlo en un caso paradigmático por John Locke[11].

Entre los aprioristas, debemos distinguir entre quienes consideran como criterio de conocimiento la intuición o (*vel*) la evidencia; aquí podemos encontrar Platón y su *noesis*, que está sobre la *dianoia*, o conocimiento lógico racional[12], que volvió prepotentemente a repuntar a la palestra en los años 90 del siglo pasado gracias a las revolucionarias propuestas de Raymond Duval[13]; pero tal vez el apriorista más paradigmático a favor de la intuición y de la evidencia fue René Descartes (1637), con su criterio de las ideas claras y distintas, que da precisamente, como paradigma de la evidencia, el famoso *cogito*.

En el otro extremo, en el de quienes sostienen el *a posteriori*, encontramos los que podríamos llamar logicistas, para quienes el raciocinio está en el escalón superior, o incluso la lógica misma. Es decir, por ejemplo, Aristóteles y Leibniz. Para Leibniz, de hecho, todo es analítico, toda verdad tiene su base en un principio lógico de identidad, por lo menos para una mente perfecta; por tanto las cosas que para nosotros son empíricas, son tales porque no las sabemos "calcular"[14], pero no son empíricas para Dios.

Aquí el innatismo toma una forma muy atenuada: podemos incluso no tener ideas innatas (como piensan, por el contrario, Platón y Descartes) pero tenemos al nacer, en cuanto ser humanos, como mínimo la *facultas*, la capacidad intelectual; ésta es la potente respuesta de Leibniz a Locke en los Nuevos *ensayos sobre el entendimiento humano*[15]: *nihil est in intellectu quod prius non fuerit in sensu*, nada está en la mente que antes no haya estado en los sentidos, *nisi intellectus ipse*, sino el intelecto mismo. En otras palabras: si la mente en el momento del nacimiento fuera una *tabula rasa*, no se vería el porqué no podrían llegar a conocer los gatos; al nacer el niño y el gato tienen una mente vacía de contenidos empíricos, pero el primero tiene la capacidad formal de organizarla según la lógica.

Debemos también hacer una segunda esencial distinción entre los racionalistas. Dado que el conocimiento tiene dos polos, el sujeto que conoce y el objeto conocido, debemos distinguir entre quienes consideran el principio activo, o criterio de verdad, en el primero y quienes lo atribuyen al segundo. Tenemos así los idealistas, en el primer caso, Fichte, Schelling, Hegel, y los

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[11] Locke (1690), pero igualmente se puede atribuir a prácticamente cualquier filósofo anglosajón (Williams & Montefiore (1966); Turco (1974).

[12] D'Amore, Fandiño Pinilla & Iori (2013: 86-87).

[13] Duval (1993; 1995).

[14] Es decir derivarlas de un *calculus ratiocinator*; véase D'Amore (2001b).

[15] Leibniz (1704).

realistas, Aristóteles, Leibniz, Espinoza. En medio está Kant: las formas *a priori* están en el sujeto, y constituyen un filtro obligado para el objeto, que no es conocido sino mediante su construcción en forma de "fenómeno"[16].

En el otro sentido, los empiristas toman una vía diversa para recuperar el saber formal, es decir el nominalismo. Para Locke las "verdades de razón" deben ser justificadas; lo cual no significa no creer en la matemática, obviamente, sino que las verdades de la matemática tienen que ver con los nombres puros, es decir, objetos inexistentes, no las cosas, las *res*. De aquí se explica la gran tradición de estudios de lógica de los anglosajones, de los *calculatores* de Cambridge hasta Boole.

Este intento personal de síntesis aparecerá obvio y burdo, pero es un esqueleto sobre el cual se puede razonar en el ámbito didáctico que es el que nos interesa. Veamos cómo.

3.1. Conocimiento y hermenéutica

El conocimiento no es una banal reduplicación del mundo, como quisiera un tipo particular de positivismo o el neo-empirismo lógico: eliminado el pensamiento, tendríamos dos mundos y no uno y, simplemente, nos hubiéremos ciertamente complicado la vida. (Personalmente fueron fundamentales en este campo para mí los estudios de las siguientes obras)[17].

Tomemos por ejemplo el conocimiento histórico. Para un positivista la historia es el pasado más el pensamiento del historiador, con este último que tiende a cero; pero así el resultado sería la crónica, no la historia. Por el contrario, el conocimiento histórico es el pasado más una hermenéutica. Por tanto, el conocimiento debe ser un esquema conceptual proyectado sobre lo real, no un segundo real, en mi opinión inútil. Esta referencia a la hermenéutica nos lleva al interior de reflexiones que ya hicimos en didáctica de la matemática[18]. En la pág. 20, Bagni escribe:

> 67. La interpretación es un momento clave del acercarse a un texto, a un contenido, por tanto para el aprendizaje; pero para interpretar es indispensable acercarse de cualquier forma al saber en juego, y esto nos lleva al círculo hermenéutico. 68. El aprendizaje es asimilable a una construcción más que a la contemplación.

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[16] Kant (1781).
[17] Santucci (1970); Pasquinelli (1969).
[18] Bagni (2009).

3.2. Conocimiento y prejuicios

El segundo tema que debemos afrontar es aquel de los *idola*, en el sentido baconiano, es decir los prejuicios[19]. Aquí me refiero básicamente al conocimiento "científico", del cual *idola* son las siguientes afirmaciones consideradas por los ingenuos como nociones comunes o axiomas:

- el conocimiento científico es cierto
- el conocimiento científico es verdadero
- el conocimiento científico es estable.

Encontrar contraejemplos es muy fácil, especialmente para un matemático. ¡Cuánto daño hacen en el proceso de enseñanza — aprendizaje estos axiomas — *idola*…!

Por tanto, de la aporía de la doble alma constituida de una parte por la aspiración a la estabilidad, y de la otra por el "progreso" (que es de signo opuesto a la certeza y a la estabilidad), propongo salir pasando a un nivel más fino, es decir al concepto de teoría; como hicieron los pluralistas (Empédocles, Anaxágoras, Demócrito,...) para salir de la contraposición Heráclito/ Parménides. Tomemos por ejemplo a Demócrito: los átomos son siempre los mismos, y aquí recupera la estabilidad eleática de Pármenides, pero se combinan continuamente en formas diversas, y aquí recupera el *panta rei* de Heráclito. De la misma forma, podemos dar certeza, verdad y estabilidad local, dentro de una teoría, pero la ciencia pasa continuamente de una teoría a otra.

3.3. Conocimiento e individuo

Un desarrollo dialéctico futuro podría estar contenido en la observación de que el conocer lleva necesariamente a la epistemología y que el conocer necesita de un discurso sobre el aprendizaje, lo cual se generaliza, tal vez, en nuestro campo, con el término "educación". No es posible afrontar este tema sin usar el término "individuo", la historia cultural que lo define, la ética que indudablemente acompaña cada una de las reflexiones. Aprender es entonces la subjetivación y la transformación debidas al aprendizaje y a la objetivación. Con la inmediata consecuencia: mientras aprendo, cambio desde cualquier punto de vista, pero soy siempre la misma persona. Y conmigo cambia quien

[19] Bacone (1620).

me enseña. Y terminamos con ser indistinguibles respecto al conocimiento. Pero este punto 3 deberá ser el tema central de un desarrollo posterior.

4. Acción, labor, praxis

Los términos que titulan este parágrafo giran alrededor de la evidencia del hecho de que, en situación de enseñanza y aprendizaje, los dos polos de la acción, digamos docente y alumno, comparten una práctica que los vincula, que los modifica, con papeles y actividades no siempre distinguibles, que se basa en la idea de *labor* entendida en el sentido marxista del término.

En su "conferencia de Barranquilla", Radford nos da las líneas guía de esta interpretación[20]. Cita justamente la *Introducción a la crítica de la economía política* de Karl Marx[21] y «las dos grandes categorías con las cuales se puede definir el trabajo:

1) las *relaciones de producción*, es decir las formas históricas y culturales de interacción humana;
2) los *modos de producción*, es decir la manera de producir de los individuos».

No vale la pena agregar algo más, reenviando necesariamente al mismo texto de Radford [perfecta la citación de Dupré que hago mía: "Ni la materia bruta ni los instrumentos constituyen la fuerza económica hasta que son integrados dentro de un sistema social"[22]].

Desde esta perspectiva histórico-social hay que notar la precisa y profunda crítica que hace Radford a la interpretación de la función del alumno, como *propietario privado* que debe *construir* su propio saber *negociando* sus significados, y del maestro que guía la construcción del alumno[23]. Comenta Radford[24]: "No hay teoría en la educación matemática que se haya apegado con más fuerza y haya promovido con tanta energía esos conceptos como el constructivismo norteamericano".

Yo quiero ir más allá, pero, eso sí, en la misma dirección.

Dado que de trabajo se trata, hay que definir un valor, recordando teorías económicas clásicas: el valor de cada cosa depende de la cantidad de trabajo

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[20] Radford (2013b).
[21] Marx (1857).
[22] Dupré (1983: 86).
[23] (Radford, 2013b).
[24] *Ibidem*.

necesario para producirla (Adam Smith, David Ricardo, Karl Marx, sólo para citar algunos pensadores).Y, para mí, este valor se basa en la eterna dialéctica entre "hombre histórico" y "hombre social".

Para buscar una contribución a esta pista de análisis, me sirvo de Friedrich Engels[25].

¿Por qué citar precisamente la edición italiana? Porque la tercera edición italiana de este extraordinario texto de epistemología de la ciencia (marcadamente dialéctica) fue editada por un personaje de excepción, Lucio Lombardo Radice, matemático muy bien conocido en el ambiente italiano, político activo en el partido comunista, con gran interés por los problemas de la enseñanza y el aprendizaje de la matemática y por la divulgación de la misma.

En la introducción de Lombardo Radice, se lee: "El interés de Marx estaba netamente polarizado hacía el *hombre histórico*, el hombre de la labor y de la producción social, hacía la dialéctica de la praxis humana social" [26].

En la pág. 17: "Marx veía en el trabajo y en la producción social un elemento del todo nuevo y original, respecto a los procesos naturales, que implica otra dialéctica. (…) El hecho es que el elemento primero, el constituyente elemental de tal dialéctica es la praxis, la actividad humana del trabajo. "El defecto principal de cada materialismo hasta hoy,… es que el objeto, lo real, lo sensible es concebido sólo bajo forma de objeto o de *intuición*; pero no como actividad humana sensible, como actividad práctica, no subjetivamente"" (la frase entre comillas internas es tomada del apartado de F. Engels al interior del libro, en el capítulo: Primera tesis sobre Feuerbach).

En la pág. 18: "Sobre la "parte obtenida de la labor en el proceso de humanización del simio" el lector encuentra, en esta *Dialéctica de la naturaleza* de Engels, un ensayo que desarrolla brillantemente y con detalle la idea de Marx, que el hombre es el resultado de su propio trabajo".

Sobre este punto se cita un artículo de otro intelectual italiano, Palmiro Togliatti; Togliatti afirmaba que "un verdadero naturalismo y un auténtico humanismo no pueden surgir sino a condición que la realización de la naturaleza humana sea entendida como el resultado de un proceso"[27].

Lo mismo en la pág. 21: "La acción recíproca excluye todo primario absoluto y todo secundario absoluto; pero es igualmente un proceso de dos

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[25] Se trata de la célebre obra *Dialéctica de la Naturaleza* que Friedrich Engels proyectó como idea para la redacción de parte de un volumen que no alcanzó a concluir, cuyo índice parece fue elaborado en 1878 una primera vez y después en 1880. Pero yo uso la edición de 1956, Roma: Editori Riuniti, por un motivo preciso que veremos. (Engels, 1956).

[26] Engels (1956: 16).

[27] Las palabras son de Lombardo Radice, p.18, nota 2, in Togliatti (1954).

caras que, por su naturaleza, puede ser considerado desde dos puntos de vista diferentes; para poder ser comprendido en su conjunto debe precisamente ser estudiado sucesivamente desde dos puntos de vista opuestos, antes que pueda ser comprendido el resultado plenamente".

Cuando decimos que la acción del docente y del alumno no son "dos acciones" sino que es "la misma acción", encontramos precisamente este punto de vista; la labor, lo que se produce, la persona que lo produce, los varios agentes, … son todos componentes al unísono de una única actividad que, con una sola palabra, podemos llamar la *labor*.

Veamos que dice el mismo Engels (1956) en la pag.190:

> Frente a todas estas creaciones, que se presentaban como productos directos de la mente y que parecían dominar las sociedades humanas, los más modestos productos de la labor manual fueron relegados en un segundo plano; […]. Todo el mérito de los rápidos progresos de la civilización fue atribuido a la mente, al desarrollo y a la actividad del cerebro; […] incluso los científicos materialistas de la escuela darwiniana no logran aún hacerse una idea del origen del hombre porque, estando aún bajo la influencia ideológica del idealismo, no reconocen la función que ha tenido el trabajo en aquel proceso.

Según mi opinión es una notable identificación que ayuda a entender posiciones diversas, de hecho antitéticas, en el proceso de la labor en aula, pero también en su interpretación, aquel "valor" sobre el cual he puesto el acento inicial.

Insisto: enseñar y aprender son indistinguibles, el docente se transforma a sí mismo en la práctica de enseñar, así como el alumno se transforma al aprender. Estas transformaciones se deben a la labor puesta en práctica por los dos de forma personal pero en el contexto social de pertenencia, el contexto escuela, en la evolución de la práctica de aula, que es una acción social (no individual) puesta en acto por todos aquellos que participan en ella.

El "objeto" construido en este trabajo está previamente delimitado, así como lo es en el trabajo en general, no se trata de replicar un modelo o construir uno de nuevo, sino de acercar el resultado de la acción a lo esperado, que alguien llama institucional[28]. No es necesario que el objeto matemático en construcción preexista de cualquier forma metafísica, identificable con el objeto de conocimiento en juego, es suficiente que este sea parte de la transposición didáctica de un objeto institucional puesta en obra por el docente[29].

[28] D'Amore & Godino (2006).
[29] D'Amore (2001c).

5. Semántica y libertad de expresión

Nunca se evidencia lo suficiente el hecho que nuestras expresiones, en cualquier contexto, frente a una aparente libertad, son por el contrario condicionadas por contextos de carácter antropológico y sociológico. Para dar fuerza a esta idea, que considero fundamental, me sirvo de los estudios de Benjamín Whorf. Los intercambios dialógicos son siempre fruto de acuerdos, a menudo implícitos; estoy pensando a ideas como “el corte de las lenguas madres”, utilizando las palabras de Benjamín Whorf.

Es mi intención ser más preciso, utilizando otra citación de Whorf :

> Todos nosotros conservamos una ilusión sobre el acto de hablar, la ilusión es que el hablar está libre de obligaciones, que es espontáneo, y que simplemente “expresa” cualquier cosa que deseemos expresar. Esta apariencia ilusoria deriva del hecho que los fenómenos obligatorios al interior del flujo aparentemente libre del discurso, son tan completamente despóticos, que el hablante y el oyente están ligados inconscientemente como por una ley natural.[30]

Y aún:

> Nosotros seleccionamos la naturaleza según líneas trazadas por nuestras lenguas madres, las categorías y los tipos que aislamos del mundo de los fenómenos no los encontramos allá porque están ahí, delante de los ojos de cada observador; por el contrario, el mundo es presentado en un caleidoscópico flujo de impresiones que debe ser organizado en nuestra mente. Nosotros hacemos a pedazos la naturaleza, la organizamos en conceptos, y esto básicamente porque participamos en un acuerdo de organizarla de tal forma; un acuerdo que es aceptado por toda nuestra comunidad lingüística y es codificado en los esquemas de nuestra lengua. El acuerdo es totalmente implícito y no declarado, pero sus términos son absolutamente obligatorios; no podemos en modo alguno hablar sino sometiéndonos a la organización y a la clasificación de los datos que el acuerdo impone. Este hecho es muy significativo para la ciencia moderna, porque significa que ningún individuo es libre de describir la naturaleza con absoluta imparcialidad, sino que es obligado a interpretarla de cierta forma incluso cuando se considera máximamente libre.[31]

A esta posición de Whorf se opuso con decisión un vasto grupo de lingüistas; entre todos señalo a Louis Hjelmslev; véase, por ejemplo, Hjelmslev[32].

Pero no profundizo aquí con esta controversia, aunque alguien debería hacerlo. Los efectos de esta contraposición en relación con los estudios en didáctica de la matemática los considero más que evidentes.

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[30] Whorf (1959).
[31] Whorf (1940).
[32] Hjelmslev (1943), en particular los capítulos 13, 15 y 21.

¿Qué sucede en este intercambio dialógico, en una situación de enseñanza — aprendizaje, asumiendo que el conocer es posible y, como proceso, dado en una práctica compartida en la cual está en acto una *labor* que crea un hombre histórico, fruto no sólo de dicha *labor*, sino también de sus orígenes culturales y sociales?

La complejidad del fenómeno es evidente.

Sería oportuno dar ejemplos; pero, en espera de análisis más profundos, específicos y ejemplificativos, me limito a hacer una pequeña referencia al complejo de prácticas compartidas que se reúnen bajo la denominación "dar un solución a una situación problemática".

Por "situación problemática" entiendo no solo un texto, no solo estrategias resolutivas, no solo conocimientos matemáticos involucrados, sino el sistema de competencias reales, específico para dicho problema, en el marco del cual se puede imaginar todo lo descrito por el significado semántico del texto que conecta con las experiencias de cada uno de los aprendices[33].

Sobre estas experiencias (aquellas que después se pondrán en acto durante la *labor*), señalo los "factores expertos" o, por lo menos, aquellos que considero que tienen un peso mayor (una lista completa es impensable):

- experiencia (esta ya abriría un mundo en sí);
- habitud a expresar (oralmente y por escrito) ideas y acciones;
- capacidad de hacerse representaciones internas;
- capacidad de proponer representaciones externas;
- competencia matemática adecuada;
- competencia lingüística adecuada;
- …

6. La práctica de aula, algunas observaciones

Podría ser interesante reflexionar sobre algunas componentes de las prácticas de aula, que siempre se citan; entre estas, el trabajo cooperativo por su valencia cognitiva fuerte (por ejemplo el trabajo en grupo de los alumnos). No se debe creer en una unicidad de interpretación terminológica ni siquiera en este caso[34].

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[33] D'Amore (1993; 2014).
[34] D'Amore & Fandiño Pinilla (2012).

Son múltiples y profundos los análisis modernos sobre esta metodología, por ejemplo los estudios que definen las *Relaciones cooperativas en la escuela*[35].

Por ejemplo, existen varias acepciones de grupo:

- acepción sociológica: el grupo es un conjunto de dos o más individuos que buscan un mismo objetivo individual; sociología, estadística y derecho identifican en la objetividad de la tarea y en la coexistencia física de cada uno de los individuos y de los subgrupos el elemento significativo, pero no toman en consideración los aspectos relacionales, de comunicación, ni las dinámicas emotivas y afectivas;
- acepción antropológica: el grupo es un conjunto de individuos que se reconocen en determinados valores, mitos, tradiciones, ceremonias, rituales, sistemas de signos; el antropólogo se interesa por la cultura y por el proceso de enculturación (es decir por la transferencia del patrimonio cultural de una generación a otra) y de aculturación (identificada en la hibridación entre culturas); aquí el individuo es tanto un usuario como un agente activo de la cultura; cada sujeto actúa, cree y ritualiza y así mantiene viva la propia cultura a la cual pertenece; además, aceptando otros contextos culturales, modifica su cultura de origen;
- acepción psicológica: el grupo es un conjunto de tres o más individuos que se reúnen como grupo y tienen entre ellos relaciones de influencia recíproca; el psicólogo centra su atención en las relaciones, las comunicaciones y por tanto en el sentido de pertenencia al grupo; se habla de grupo sólo cuando se establecen, gracias a retroalimentación, relaciones circulares; aquí se estudia con mucha atención la relación entre emitente y receptor;
- acepción analítica: el grupo es un conjunto de tres o más individuos que comunican interactuando entre ellos según una matriz común interpersonal, según un sentir y un pensar progresivamente compartido que se convierte en patrimonio del grupo; en esta acepción es necesaria una matriz de grupo en el cual las comunicaciones interpersonales transcienden el individuo; lo que aquí interesa es la formación de un pensamiento compartido;
- acepción pedagógica: un grupo es un conjunto de sujetos — personas que comparten contextos y relaciones dirigidas a reconocer y promover las potencialidades individuales en las diferentes edades de la vida; se trata de una de las acepciones más cercanas a la que nos interesa; pero, según

[35] Dozza (2006).

la pedagogía, todos los aspectos precedentes deben ser valorizados dado que cada uno de estos contribuye a crear la identidad misma del grupo y a estudiar las dinámicas que lo caracterizan; la pedagogía toma en consideración todos estos aportes con el fin de crear una significativa planificación y una reflexión entre los miembros del grupo para alcanzar una expansión, enriquecimiento, realización de sí y en especial como instrumento de orientación hacia el futuro; en este sentido, la formación pedagógica reconduce los valores al sujeto-individuo- hombre-persona y a su constituirse como tal; para formar la base de esta constitución se encuentran valores diversos, como los principios fundamentales de libertad y de igualdad, el reconocimiento y la valorización de las diversidades propia y la de los otros; todo esto constituye la base de un proyecto existencial no sólo para cada uno de los individuos sino para toda la sociedad, sobre la base de la convivencia democrática y de una emancipación individual; las condiciones de esta convivencia radican en la apertura crítica sobre sí mismo, de tipo racional, anti-dogmática; notable la colaboración de la pedagogía con las otras ciencias;

- acepción formativa: el grupo es un conjunto de dos o más sujetos-personas que establecen relaciones de interdependencia y coordinan sus acciones y comunicaciones en contextos específicos con el fin de perseguir el aprendizaje y la co-construcción de identidades, inteligencias y significados; es esta la acepción que mayor relación tiene con la didáctica; la atención aquí se centra en el currículo formativo y sobre las acciones, relaciones, comunicaciones, construcciones y re-construcciones de los conocimientos a nivel intra- e inter-subjetivo; los estudiosos de esta acepción observan la organización de los contextos de aprendizaje y de formación, la inter-dependencia y la responsabilidad individual al interior del grupo, el dominio de cada una de las competencias sociales y el ejercicio de las habilidades lógicas, la reconstrucción personal de conocimientos y de competencias, la motivación intrínseca y la capacidad de considerar el sentido constructivo del error; la capacidad de reflexionar sobre la experiencia vivida es uno de los pilares del grupo de trabajo lo cual permite una continua planificación que debe llevar al empeño de cada uno de los sujetos involucrados que intervienen en el proceso de enseñanza y de aprendizaje; todo esto se presenta dando gran importancia a la continua re-definición de los contenidos, de los contextos y de los procesos cognitivos y emotivos.

Como se ve, definir que es un grupo, que significa *labor* en una práctica compartida, es problemático y complejo, pero se han dado grandes avances, respecto a las primeras apariciones de esta metodología que aparecía un poco confusa e ingenua. Hoy todo es claro, todo es categorizado y formalizado, y se basa sobre el concepto de trabajo realizado en común.

Quiero concluir recordando la metodología didáctica de la discusión en aula, en la cual el grupo coincide con la clase; se trata de un óptimo momento de atribución de significados personales y compartidos y de conceptos entre docentes y alumnos y entre alumnos, que tuvo precisamente en la didáctica de la matemática extraordinarios éxitos.

Señalo también que el estudio de las comunidades de práctica que desarrollan matemática, y por tanto principalmente las clases, fueron tomadas en seria consideración en los último años como verdaderos grupos sociales, usando como instrumento la sociología, con resultados de gran interés tanto teórico como práctico[36].

7. Conclusión

Creo haber creado más problemas que cuantos haya intentado resolver, pero me he sentido en el deber de asumir algunos términos usados recientemente por Luís Radford, para confrontarlos con puntos de vista personales, incluso para reafirmar su complejidad.

Sobre estas bases, considero poder mostrar que no existen contradicciones entre la teoría de la objetivación y la idea misma de situación, tal como se presenta en la teoría de las situaciones didácticas. Siempre he sido un fanático defensor de la unificación de teorías[37], más que de su proliferación; a veces las crisis, las diferencias, las rupturas se dan o, mejor, se evidencian, porque no se tiene la paciencia de buscar las raíces últimas, verdaderas, reales de cada una de las teorías. En la base de la objetivación se encuentran raíces culturales e históricas de los individuos y de las teorías; pero en la base de la definición de las situaciones elaborada por Guy Brousseau en los años 70 del siglo pasado también existen raíces epistemológicas y culturales que debemos respetar y tener en cuenta. En más de una ocasión, Brousseau llama la atención sobre la "inmersión en las didácticas específicas de los diferentes conocimientos"[38] poniendo en evidencia el papel de las raíces culturales. En el fondo, además, muchas de sus reflexiones sobre los obstáculos epistemológicos

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[36] Bagni & D'Amore (2005); D'Amore (2005); D'Amore & Godino (2006; 2007); D'Amore Font & Godino (2007; 2008).
[37] Prediger, Bikner-Ahsbahs & Arzarello (2008); Radford (2008).
[38] Brousseau (2008: 108).

no son otra cosa que el análisis cultural de los conocimientos que determinan el trabajo en el aula.

Pero el discurso se hace complejo y mérita un futuro estudio especifico centrado en este preciso argumento.

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After the fall of the Babel Tower: The predicament of the researcher in the field of mathematics education

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Much has been said in the last few years about the current fragmentation, not to say pulverization, of research in mathematics education; or, to use a different metaphor, about the plurality of voices, not always harmonious, that became almost the hallmark of the field. It is as if we lived the consequences of God's decision to confound the researchers' languages in the midst of their vigorous attempts to erect the Tower of Babel.

For some time now, mathematics education researchers have been documenting and analyzing this phenomenon. Lerman (2010) and Jablonka & Bergsten (2010), for instance, asked whether plurality is a problem, and Gellert (2010), using Luis Radford's (2008) conceptualization of the notion of theory, offered advice about how to "integrate theories in mathematics education", at least "locally". In this paper, I wish to revisit the issue of the extreme "multivocality" from a perspective somewhat different than the one that has been dominating the discussion so far. Until now, the discussants have been drawing a topological map of the entire field, delineating its multiple towers and diverse inter-tower communication problems. The debate, like the biblical tale, has been mainly about the activity of the community as a whole and about its collective fate. This treatment, though, has been too abstract and detached to help the individual tower builder in her predicament. And predicament it is. After the discontinuation of the Tower of Babel project, the builder faces great many dilemmas.

I have been experiencing these problems for many years now, but never took the time to fuss about them. I also knew it was not just my impression:

Anna Sfard, "After the Fall of the Babel Tower:
The Predicament of the Researcher in the Field of Mathematics Education"
in Laura Branchetti, *Teaching and Learning Mathematics.*
Some Past and Current Approaches to Mathematics Education, pp. 173-184.

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my observations that academia has been changing, and that the resulting dilemmas were increasingly difficult to cope with have been corroborated time and again by testimonies of colleagues from different parts of the world. Whether young or not so young anymore, whether experienced or novices, whether from Asia, Europe or Africa, each one of us seemed to be asking the same question: In times when every mathematics education researcher tends to hold to her or his "signature" discourse and when not many of them do their best to make themselves clear, how am I to communicate with others, build on work already done, and at the same time stay true to myself, to my own discourse? As time went by and the frequency of these and similar queries have been growing exponentially, I realized that the familiar problem which I have learned to always leave "for later" could no longer be pushed aside. It was now a matter of life or death, a matter of the researcher's survival in post-Babel times of the thwarted dreams of the single tower, high enough to gratify the mortals with God-eye view of the world. I understood that fussing about the fate of the post-Babel tower builder was not just an option anymore — it became imperative.

To initiate the debate, I frame the main part of this essay as a conversation between a young academic, fresh from her doctoral studies, whom I will call Dr. Young and her old mentor, Prof. Sage. The young person has many pressing questions, and the professor, from the height of his years and experience, is expected to help her in finding some answers. Although the protagonists are not to be equated with any real persons, the conversation between them is drawing on my conversations between real people. And one more thing: Although Dr. Young and Prof. Sage do not agree on many matters, they do share a discourse. I precede their conversation with a brief explanation on how they use the word research and its cognates.

1. Speaking about research: The glossary

As researchers, Dr. Young and Prof. Sage see themselves as storytellers[1]; or, combining the metaphors, as builders of towers of stories. Their narratives

[1] I am using here the terms *story* and *narrative* as synonyms, referring to any sequence of interrelated utterances about a given set of objects. The utterances may be interconnected either chronologically, that is, according to events reported in the story; or logically, that is, according to relations such as inference, negation, etc. Note that traditionally, only the first type is considered as narrative. For instance, Bruner (1986) made the distinction between narrative and paradigmatic mode of thinking, and this distinction parallels the present one, with chronologically organized stories corresponding to the narrative mode and logically organized ones to the paradigmatic. Elements of both these types can be found in most stories

have distinct characteristics, and only those stories would do that can mediate human activities and make a tangible, desirable difference, at least in principle. There is, of course, the weighty question of who decides what kind of difference should count as "desirable" and how such judgment can be carried out, but this must be left for another occasion. For now, it suffices to say that the impact may express itself as an increase in the effectiveness and productivity of human activities or in these activities' complete restructuring. In the case of researcher in mathematics education, the activities supposed to benefit from researchers' narratives are those of teaching and learning mathematics.

The usefulness of the researcher's stories, their capacity for making a real difference, depends, of course, on how well they seem to fit our experience in the world. A story that presents a person as able to walk through the wall, although potentially appealing to a filmmaker, has little chance to be endorsed by a researcher; as does the story that portrays learning mathematics as happening only when the student falls asleep the moment she enters the classroom. Research narratives must also be more comprehensible to their users than the instructions of one Babel Tower builder in the ears of another. To ensure the required qualities, researchers subject their storytelling to rules much stricter and more explicit than those that govern telling stories in everyday life. These rules define research discourse, which is recognizable by its specialized vocabulary; its visual mediators (to use example coming from science, think about Einstein's most famous story told symbolically as $E=mc^2$); its routines of constructing and endorsing narratives, and the resulting narratives themselves.

The name theory will be reserved for any consistent, logically interconnected set of endorsed narratives produced within a given discourse. "Logically interconnected" is probably what Bernstein had in mind while speaking about vertically organized knowledge[2]. In any case, the underlying idea is that the set of stories called theory can be organized into a hierarchy of layers, so that narratives at any given level are endorsable only if those at the preceding levels have already been endorsed.

From these definitions, discourse and theory emerge as tightly related, but distinct. Whereas discourses provide keywords and rules for these words' uses, theories follow these rules when using the words in their stories. By defining words and offering rules for constructing and endorsing narratives,

people tell. Nevertheless, it is still possible to segregate these stories into chronologically — organized and logically — organized according to the leading organizational principle. Traditionally, research stories have been required to be of the latter, logically organized type.

[2] Bernstein (1999).

discourses lay epistemological and ontological foundations on which the towers of stories, a.k.a. theories, are erected. Since they often do this fundamental work only implicitly, Foucault's term "epistemological unconscious"[3] may be a better metaphor than "foundations". Whereas there is no possibility of a fully-fledged theory without a well-defined discourse or, to use Basil Bernstein's term, without a discourse with a strong grammar[4], there may be discourses that thrive and gain popularity without producing well-formed theories. Using Bernstein's language, these latter discourses can be said to have weaker grammars and to produce horizontally organized sets of endorsed narratives[5]. The distinction between discourse and theory is rarely present in mathematics education debates [6], but it is crucial for the researchers' ability to deal with the dilemmas that proliferate in post-Babel reality.

The discursive vision of research presented here and Bernstein's theory of codes resonate with the postmodern philosophers' declarations that knowledge is "a kind of discourse[7]" or a "conversation of mankind[8]". In equating research with discursive activity one also comes close to Michel Foucault, for whom discourse, or better still discursive formation, means "things said… [n]ot books…, not theories…, but those familiar yet enigmatic groups of statements that are known as medicine, political economy, and biology"[9]. Both mathematics and research in mathematics education fall under this definition, and both can thus be seen as discursive activities. Recursively, research in mathematics education becomes the activity of telling stories about people telling stories about mathematical objects. Concurring with Foucault

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[3] Foucault (1970-1994).

[4] Bernstein (2000).

[5] Which, as said before, do not count as theories.

[6] Let me remark that if Basil Bernstein made this distinction, he would have probably revised his statement that mathematics is "horizontally" organized, that is, grows by putting one new tower next to, rather than on the top of the other. He would have then to admit that mathematical stories are organized hierarchically, thus vertically. Although there are many seemingly unrelated mathematical theories, all of them sit within one broad discourse, defined by a relatively homogeneous vocabulary and routines. Moreover, mathematical discourse develops by swallowing its successive meta-discourses, which results in a distinctly hierarchical structure of its theories. Add to this mathematicians' unrelenting effort to unify their theories (this was, for example, Bourbaki's explicit goal), and if you interpret fields as discourses and towers as theories, you begin seeing mathematics as a paradigmatic case of tightly interconnected vertical field. If Bernstein saw mathematics otherwise, he might well be a graduate of a typical, and typically disconnected, mathematics curriculum!

[7] Lyotard (1979: 3).

[8] Rorty (1979: 389).

[9] Foucault (1972: blurb).

and with his interests, I am concerned in this paper with the plurality of discourses rather than of theories. And the reason is simple: these are the distinct forms of communication called discourses that bear the main responsibility for partitioning research in mathematics education into multiple communities.

All this talk about research as discursive activity sounds good so long as research is seen from a historical-collective perspective. But the declared focus of this text is on activities and dilemmas of an individual researcher. In consequence, the question must be asked of whether the claim that psychology, sociology or mathematics are communicational activities exhausts all that has to be said when one considers an individual engaged in psychological or mathematical research. Most people would say that in the case of research as an activity of a person, communication is just a part of the story, one that comes relatively late in the plot. They would insist that there is another, in a sense more important element of the researcher's activity, called thinking. Since thinking is traditionally equated with manipulating mental entities, and this entails its being ontologically distinct from communicating, the claim that the term discourse suffices to deal with research as an activity of an individual does not seem to hold. Fortunately, at least for the simplicity of our conversation, this drawback disappears when one realizes, admittedly with some help from such powerful thinkers as Vygotsky and Wittgenstein, that we do not, in fact, need to have recourse to mental entities to understand thinking. Vygotsky's psychology[10], when combined with Wittgenstein's philosophy[11], inevitably leads to the conclusion that thinking can be usefully defined as a form of communication[12]. The resulting demise of the thinking-communicating dichotomy ensures that the discursive description of research is exhaustive and that the definition of research as storytelling holds on both historical — collective and ontogenetic — individual levels. For Dr. Young and Prof. Sage, this definition also entails certain well-defined commitments that one inevitably makes, whether explicitly or only implicitly, when assuming the role of researcher.

It is time to switch from the soliloquy to dialogue. From the conversation that follows, the post-Babel tower-builder's problems will emerge as a matter of inherent tensions between the builders' diverse *dos* and *don'ts*, those related to their obligations to their discourse, to their colleagues, and to themselves.

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[10] Vygotsky (1987).
[11] Wittgenstein (1953).
[12] Sfard (2008).

2. The sorrows of the young mathematics education researcher – a conversation

Prof. Sage: Good to see you again, Emy — or should I now call you Dr. Young? In any case, how are you these days?

Dr. Young: Not so good, I'm afraid. I always thought that after I complete my PhD, my life would finally begin. Well, nothing of the sort. It only gets harder. See, I am trying to write. I thought that once I made it through the hurdle of dissertation, I would be doing nothing else, but telling stories from classrooms, analyzing transcripts, sharing my insights. But I never get to this. Instead, I am wasting my time reviewing literature and explaining my terminology. Why do I have to do it time and time again?

Prof. Sage: What you say sounds familiar and familiarly annoying. Like you, I wish we had no need for explaining our vocabularies. I wish we could say "Researchers of the world, unite". I would like God to be able to say about the research community what she said about pre-Babel people: "nothing will be withheld from them which they purpose to do". As seems to be the case in mathematics, and as it had been in science, at least until not long ago. And yet for this, we would have all to speak the same language. In our post-Babel times, this is certainly not the case. Ever since Kuhn and his revolutionary insights about scientific revolutions (pun intended) almost nobody seems to believe in the possibility of a single discourse superseding all the others. Or wishes for this, for that matter. So let me tell you what I am telling myself over and over again: you have to explain your vocabulary because other folks don't live in your tower — they have their own.

Dr. Young: If so, how do you explain the fact that so many authors do not define their keywords? Aren't we expecting too much of ourselves? Why should we be different?

Prof. Sage: I cannot talk for these other folks; I can only testify for myself: Defining is something I do for my own good. I do this to make sure that I am understood the way I want to be. With all those different discourses floating around, there is no certainty that people who are using the same words are talking about the same things. But c'mon, can you really say you don't need authors' explanations about how they use words?

Dr. Young: Well, sometimes I do, sometimes I don't. I certainly don't need the author to tell me what learning is. Or mediation, or scaffolding, or number sense, or identity. Today, these are everybody's words. When authors try to explain words every child would understand, I feel patronized. Besides, defining is recursive and there is no end to it.

Prof. Sage: Wait, you said number sense? Every child knows what number sense is? Shall I remind you what happened when a journal editor required you to strengthen your claims on the basis of research on number sense done by several other authors? "How can I build on it", you wondered, "if I cannot tell what the authors meant by number sense or even whether all of them were referring to the same thing?" You also remarked that some of these other authors were trying to define all right, but their definitions were not really helpful. And I know only too well what you are talking about. In my language, many of the definitions in our professional literature are not operational — they don't really tell you how to decide whether the word is applicable or not[13].

Dr. Young: But can you really operationalize everything? Look at how many people tried to define 'number sense', 'identity', 'emotion', even 'understanding'. If they did not succeed, why would I? Besides, our business is in human studies, not in mathematics. Haven't you said yourself that our vocabularies come from metaphors? You define — and you spoil. The very attempt to define — isn't it as if you were cuffing yourself? Your hands are now tied. Forget imagination, forget creativity…

Prof. Sage: If you are researcher rather than poet, your words are likely to make a tangible, real-life difference. If you don't like making yourself understood clearly enough, you may involuntarily cause harm. True, the "clear enough" in human sciences will never be as close to perfect as it is in mathematics, but we can still always strive for "clearer". When the author whom you try to read does not tell you what her words mean, you may try to

[13] The word "operational" is used here in the sense of Herbert Blumer (who preferred the term "satisfactory"): a term or concept is considered to be operational if (1) point clearly to the individual instances to which it refers, (2) distinguishes clearly this class of objects from other related classes of objects, and (3) enables the development of cumulative knowledge of the class of objects to which it refers (Blumer, 1969: 21). Blumer's example of a notion that had been used in literature without a sufficient operationalization was that of *attitude*. Similar criticism has been voiced by Geertz (1973) with regard to *belief* and by Sfard & Prusak (2005) with respect to *identity*.

tease this out from the texts themselves. Except that more often than not, this effort would elicit an even tougher new problem: you will realize that your discourse and the one you are trying to fathom are incommensurable.

Dr. Young: Incommensurable… Is there such a thing? Yeah, Kuhn delighted in the word, but can't we satisfy ourselves with just incompatible? Things simply contradict each other, so they cannot live one alongside the other. One of them must go.

Prof. Sage: No, no. This is exactly the point. If there is so much doubts around the idea of incommensuarability, it is because Kuhn (1962) himself was somehow unclear about it: he never really defined the notion and in some contexts, he used the adjectives incommensurable and incompatible as if they were fully interchangeable (as an aside, one could not ask for a better story to support the call for operationalization!). But Rorty (1979) has straightened things out for him. According to Rorty — and I wholeheartedly adopt his definitions — incommensurable does not mean excluding one another. Incommensurable means such that cannot "be brought under a set of rules which will tell us how rational agreement can be reached on what would settle the issue on every point where statements seem to conflict" (p. 316). Or, to put it simpler, there is no common measure that may be used to decide which is greater or better or truer. With Rorty, I will also conclude that "[i]ncommensurability entails irreducibility, but not incompatibility" (p. 388). So you encounter incommensurability when you have two discourses that use the same words in different ways and are governed by different rules. In result, the two discourses produce seemingly contradictory stories. The phenomenon is rather well known in the tower-building business. You build, and your tower grows. From time to time, you also make renovations, reorganize things. One day you climb on the top of the tower and the earth that so far appeared to be flat, now seems spherical. But try to convince those who live in those towers that still show only the flatness! Or, for that matter, to convince yourself that the sphere you now see so clearly is more than an illusion.

Dr. Young: Fine, I begin getting the idea. But can you give me an example of incommensurability of research discourses and of its consequences?

Prof. Sage: Okey. So… A few years ago I was asked to comment on a set of four studies guided by the theory of conceptual change. They were all focusing on regularly observed, systematically non-standard — and some of them would say “misconceived” — ways in which mathematics learners often solve certain types of problems with rational, real or infinite numbers (Sfard, 2007). Since none of the authors explained their use of words, and since without understanding the terms I could not learn from their studies, I began by operationalizing their focal notion of concept. Unable to tease out such definition from the four texts, I decided to simply use the one I constructed myself by combining Vygotsky’s claim that concept is a word plus its meaning, with Wittgenstein’s idea of word meaning as the use of the word in language. From now on, I would understand concept as word together with its use. I was convinced that this definition paved my way toward learning from the conceptual-change folks’ stories. Little did I know. Since in theories, concepts constitute well-organized, tightly interconnected systems, operationalization of one of them affects all the others. In a kind of chain reaction, my definition of concept evidently changed the rest of the authors’ vocabulary. I ended up realizing that I am facing research that grows from a discourse incommensurable with my own.

Dr. Young: But this means that I, who happen to think in discursive terms, have no chance to ever collaborate with this conceptual change fellow, right? Unless I move to his tower, of course. But how can I do this? My present research discourse is the discourse of my thoughts, so abandoning it would mean… well, compromising for rituals, for doing things for others rather than for myself. That my partner would like to move with me to my tower is equally unlikely, and for similar reasons. He had even more time than I to turn this discourse into a part of his identity. Combining our two discourses into one is not a possibility either — the discourses are incommensurable and the two of us are using the same words in different ways. So what? All doors closed? Am I a prisoner of my own tower, doomed for the exclusive company of its other inhabitants?

Prof. Sage: This sounds too pessimistic... There sure are ways to collaborate with other towers without ever forsaking your own. For instance, you and your incommensurable partner can launch a joint project, in which each one of you ensures her or his own niche. While he conducts an experimental study and does conceptual change analyses, you collect your classroom observations or interview some of the participant. You will then analyze transcripts using your own methods. Sometimes, the two of you may even be using the same data, provided you agree to collect them in ways that are acceptable to you both. But to make this scenario work, you and your partner will have to construct a common linguistic platform — something close to a disciplined colloquial language — to which you will have to return for solution whenever facing a problem.

Dr. Young: Easier said than done, don't you think?

Prof. Sage: I agree. And there is more. To be frank, those ideas on how to collaborate across discourses may be helpful, but I don't see them as a satisfactory solution to the problem. In fact, I still have more questions than answers. Things have changed a lot since the time I opted for academia as a place in which to spend my professional life. The rules of the game are now quite different, and some of them clash with each other. We spoke about the contradiction between the right to living in one's own tower, and the principle of collaborating and networking, both of which seem to be the name of the current academic game. But there is more, much more. Discrepancies proliferate. In result, young people like yourself are doomed to the life of uncertainties and constant tensions. My heart is with you, my young colleague. For once, I feel fortunate to have been born so many years before you, when he task of crafting useful stories about the world — whether in mathematics education or in any other domain of study — seemed pretty straightforward in comparison to how it is today. Problems, quandaries and dilemmas are almost the only provisions with which you are beginning your professional life. I hope some happy endings to at least some of these difficulties are already on their way, even if, at the moment, we are unable to see them. Perhaps our conversation helps in paving the route.

Note

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Today mathematics education arouses the interest of various subjects: not only insiders, such as mathematicians, teachers, pedagogists, philosophers, and psychologists, but also policy makers and entrepreneurs. Although the debate on mathematics education is lively and multi-faceted, it is often its socio-economic impact, more than other aspects, that comes under the spotlight. Mathematical competence, problem solving, mathematical skills required for technological development and in everyday life are indeed the keywords of mathematics education today. However, looking into the past or to the future, other issues and new challenges are brought into light, and we are led to recognize some surprising resemblances between the approaches in different historical periods or even, *mutatis mutandis*, some common structures or similar concerns. The contributions collected in this volume present different perspectives (past and present) on mathematics education. Some contemporary theories are taken into account and analyzed in order to bring out and clarify the key points of the current debates and stimulate discussion in view of future developments. Since the topic is complex and involves different disciplinary areas, we have invited mathematicians, historians of mathematics, psychologists, philosophers and researchers concerned with these issues to lead us in the exploration of the different aspects of mathematics education.

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